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APPROXIMATIONS HYPERBOLIQUES DES ÉQUATIONS DE NAVIER-STOKES.

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Résumé

Dans cette thèse, nous nous intéressons à deux approximations hyperboliques des équations de Navier-Stokes incompressibles en dimensions 2 et 3 d'espace. Dans un premier temps, on considère une perturbation hyperbolique de l'équation de la chaleur, introduite par Cattaneo en 1949, pour remédier au paradoxe de la propagation instantanée de cette équation. En 2004, Brenier, Natalini et Puel remarquent que la même perturbation, qui consiste à rajouter $\varepsilon\partial_{tt}$ à l'équation, intervient en relaxant les équations d'Euler. En dimension 2, Les auteurs montrent que, pour des sonnées régulières et sous certaines hypothèses de petitesse, la solution globale de la perturbation converge vers l'unique solution globale de (NS). En 2007, Paicu et Raugel améliorent les résultats de [BNP] en étendant la théorie à la dimension 3 et en prenant des données beaucoup moins régulières. Nous avons obtenu des résultats de convergence, avec données de régularité quasi-critique, qui complètent et prolongent ceux de [BNP] et [PR].

La seconde approximation que l'on considère est un nouveau modèle hyperbolique à vitesse de propagation finie. Ce modèle est obtenu en pénalisant la contrainte d'incompressibilité dans la perturbation de cattaneo. Nous prouvons que les résultats d'existence globale et de convergence du précédent modèle sont encore vérifiés pour celui-ci.

Abstract

In this work, we are interested in two hyperbolic approximations of the 2D and 3D Navier-Stokes equations. The first model we consider comes from Cattaneo's hyperbolic perturbation of the heat equation to obtain a finite speed of propagation equation. Brenier, Natalini and Puel studied the same perturbation as a relaxed version of the 2D Euler equations and proved that the solution to this relaxation converges towards the solution to (NS) with smooth data, provided some smallness assumptions. Later, Paicu and Raugel improved their results, extending the theory to the 3D setting and requiring significantly less regular data. Following [BNP] and [PR], we prove global existence and convergence results with quasi-critical regularity assumptions on the initial data.

In the second part, we introduce a new hyperbolic model with finite speed of propagation, obtained by penalizing the incompressibility constraint in Cattaneo's perturbation. We prove that the same global existence and convergence results hold for this model as well as for the first one.

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Chapitre 1

Introduction

1.1 Présentation des équations

1.1.1 Historique et modélisation

Dans cette thèse, nous nous intéressons aux équations de Navier-Stokes incompressibles qui régissent le mouvement des fluides newtoniens visqueux, homogènes (c'est-à-dire de densité constante) et incompressibles. Notons que les fluides usuels, comme l'air et l'eau, sont souvent supposés newtoniens, dans les conditions normales de température et de pression. Nous supposerons que les fluides considérés dans la suite remplissent tout l'espace, en dimension 2 ou 3, et ne sont soumis à aucune force extérieure. Dans ce cas, les équations de Navier-Stokes s'écrivent :

$$(NS) \quad \left\{ \begin{array}{rcl} \partial_t v(t,x) - \nu \Delta v(t,x) & = & -(v.\nabla)v(t,x) - \nabla p(t,x) \\ \operatorname{div} v(t,x) & = & 0 \end{array} \right. ,$$

où $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. Dans la suite, la dimension n sera égale à 2 ou 3. Les inconnues de (NS) sont le champ de vitesse du fluide $v : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ et sa pression $p : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$. Le coefficient $\nu \geq 0$ est appelé viscosité du fluide. Si $\nu = 0$, on dit que le fluide est parfait et il est alors gouverné par les équations d'Euler

$$(\mathcal{E}) \quad \left\{ \begin{array}{rcl} \partial_t v & = & -(v.\nabla)v - \nabla p \\ \operatorname{div} v & = & 0 \end{array} \right. .$$

Dans cette thèse, nous ne considérons que des fluides visqueux. Le coefficient ν sera donc toujours strictement positif.

Pour commencer, expliquons brièvement le cheminement qui a mené à ces équations fondamentales de la mécanique des fluides. Pour répondre à un problème mathématique proposé par l'Académie des Sciences de Berlin en 1748, d'Alembert tente de déterminer la résistance qu'oppose un fluide à un corps solide plongé dedans. Dans son travail, il a notamment utilisé la notion de dérivée partielle pour étudier le problème en dimension 2 d'espace mais il n'a pas réussi à exprimer la force due à la pression du fluide.

Plus tard, en 1755, Euler introduit le sytème $(\mathcal{E})$ qui tient compte de la pression, contrairement au travail de d'Alembert. Mais celui-ci remarque cependant que les équations d'Euler montrent que le fluide n'oppose aucune résistance si l'on y plonge un corps solide, ce qui est contraire à l'intuition.

En effet, au niveau microscopique, il existe des frottements entre les molécules du fluide. Ces frottements dissipent de l'énergie sous forme de chaleur. Si θ est la température du fluide, rappelons que son évolution au cours du temps est dictée par l'équation de la chaleur

$$D_t\theta - \Delta\theta = 0,$$

où $D_t\theta = \partial_t\theta + (v.\nabla)\theta$ et v est la vitesse du fluide. $D_t\theta$ est appelée *dérivée matérielle* de θ.

Au $XIX^{\text{ème}}$ siècle, Navier et Stokes ont l'idée d'ajouter le terme Δv responsable de la dissipation de l'énergie aux équations d'Euler, obtenant ainsi pour la première fois les équations de Navier-Stokes

$$(NS) \quad \left\{ \begin{array}{rcl} \partial_t v - \nu\Delta v & = & -(v.\nabla)v - \nabla p \\ \operatorname{div} v & = & 0 \end{array} \right.$$

qui décrivent le mouvement des fluides visqueux incompressibles.

Nous allons maintenant nous intéresser à l'interprétation de ces équations. Pour les détails et les justifications, on peut par exemple se référer aux livres de Bertozzi et Majda [3], Chemin [11], Constantin et Foiaş [14], Lemarié-Rieusset [28, 29] ou encore Temam [38].

La première équation de (NS) correspond à la conservation de la quantité de mouvement et s'obtient en appliquant le *Principe Fondamental de la Dynamique* ou seconde loi de Newton qui consiste à dire que la dérivée de la quantité de mouvement égale la somme des forces qui s'appliquent au système. Dans notre cas, la dérivée de la quantité de mouvement est égale à la dérivée matérielle

$$D_t v := \partial_t v + (v.\nabla)v.$$

Le terme bilinéaire, à valeurs dans $\mathbb{R}^n$, représente la dérivée convective du champ de vitesse et est défini comme suit :

$$(v.\nabla)v := \sum_{i=1}^{n} v_i \partial_i v.$$

Dans le cas où $\operatorname{div} v := \sum_{i=1}^{n} \partial_i v_i = 0$, il est intéressant de remarquer que l'on a

$$(v.\nabla)v = \sum_{i=1}^{n} v_i\, \partial_i v = \sum_{i=1}^{n} \partial_i (v_i v) = \nabla : v \otimes v.$$

Si, de plus, la fonction v est assez régulière, si $v \in H^{\frac{n}{6}+\frac{1}{3}}(\mathbb{R}^n)^n$ par exemple, on a

$$\int_{\mathbb{R}^n} v\,(v.\nabla)v \,\mathrm{d}x = 0. \tag{1.1}$$

Les forces internes exercées par le fluide sont la force liée à la pression $f_{\text{pression}} = -\nabla p$ et le frottement dû à la viscosité du fluide, qui traduit sa résistance à la déformation $f_{\text{viscosité}} = \nu \Delta v$, où $\nu > 0$. Le principe fondamental de la dynamique donne alors la relation suivante :

$$D_t v = \partial_t v + (v.\nabla)v = -\nabla p + \nu \Delta v.$$

La deuxième équation de (NS) traduit l'incompressibilité du fluide.

Les équations de Navier-Stokes apparaissent, seules ou couplées avec d'autres équations, dans de nombreux domaines d'ingénierie et de recherche, comme l'étude des écoulements d'eau dans un tuyau, la modélisation des courants océaniques, la conception d'avions ou de voitures, la météorologie, la cosmologie et la physique des plasmas. Par exemple, couplées avec les équations de Maxwell, les équations de Navier-Stokes constituent les équations de la MHD (ou *magnéto-hydrodynamique*) qui sont très largement étudiées. Par ailleurs, notons que le système (NS), comme d'autres équations de la mécanique des fluides (dans des milieux continus), intervient comme limite hydrodynamique d'équations cinétiques. Pour plus de détails à ce sujet, nous pourrons nous référer aux travaux de Golse [19], Saint-Raymond [37] et Villani [39].

Pour finir, remarquons que l'on peut toujours se ramener au cas où la viscosité vaut 1 dans les équations de Navier-Stokes ; en effet, considérons le changement d'échelle en temps

$$v_\lambda(t,x) := \lambda v(\lambda t, x) \ , \quad p_\lambda(t,x) := \lambda^2 p(\lambda t, x).$$

Alors le système (NS) s'écrit

$$\left\{\begin{array}{rcl} \partial_t v_\lambda - \nu\lambda\Delta v_\lambda & = & -(v_\lambda.\nabla)v_\lambda - \nabla p_\lambda \\ \operatorname{div} v_\lambda & = & 0 \end{array}\right. .$$

On choisit donc $\nu\lambda = 1$ et, pour alléger les notations, on désignera dorénavant par (NS) le système d'équations suivant :

$$(NS) \quad \left\{\begin{array}{rcl} \partial_t v - \Delta v & = & -(v.\nabla)v - \nabla p \\ \operatorname{div} v & = & 0 \end{array}\right. .$$

1.1.2 Résolution mathématique

Considérons le problème de Cauchy suivant :

$$\left\{\begin{array}{rcl} \partial_t v - \Delta v & = & -(v.\nabla)v - \nabla p \\ \operatorname{div} v & = & 0 \\ v|_{t=0} & = & v_0 \end{array}\right. . \tag{1.2}$$

On peut d'abord remarquer que, puisque $\operatorname{div} v = 0$, on peut calculer la pression p à partir du champ de vitesse v ; en effet, en appliquant l'opérateur div à la première équation de (NS) et en remarquant que (grâce au fait que l'on se place dans tout l'espace $\mathbb{R}^n$)

$$[\operatorname{div}, \partial_t] = [\operatorname{div}, \Delta] = 0,$$

on obtient l'expression suivante pour la pression :

$$p = \Delta^{-1} \sum_{i,j=1}^{n} \partial_i \partial_j \big(v_i v_j\big).$$

On peut donc considérer que le champ de vitesse est la seule inconnue dans ce système. En remplaçant la pression par son expression en fonction du champ de vitesse dans la première équation de (NS), on obtient

$$\partial_t v - \Delta v = -\mathbb{P}(v.\nabla)v, \tag{1.3}$$

où $\mathbb{P}$ est le projecteur de Leray défini par $\mathbb{P} := Id - \nabla\Delta^{-1}\operatorname{div}$. Notons que $\mathbb{P}$ est un opérateur non-local qui envoie les champs $L^2(\mathbb{R}^n)$ dans l'espace

$$L^2_\sigma(\mathbb{R}^n) := \big\{u \in L^2(\mathbb{R}^n) : \operatorname{div} u = 0\big\}.$$

On cherche donc à résoudre l'équation (1.3), que l'on confondra souvent avec la première équation de (NS). Pour commencer, nous allons nous intéresser à la conservation de l'énergie dans le système de Navier-Stokes.

Prenons v solution de (NS) et supposons-la dans $L^3_T\dot{H}^{\frac{n}{6}+\frac{1}{3}}(\mathbb{R}^n)^n$, c'est-à-dire $v \in L^3_T\dot{H}^{\frac{2}{3}}(\mathbb{R}^2)^2$ ou $v \in L^3_T\dot{H}^{\frac{5}{6}}(\mathbb{R}^3)^3$. En prenant le produit scalaire L^2 entre la première équation de (NS) et v, on obtient

$$\big(\partial_t v, v\big)_{L^2} - \big(\Delta v, v\big)_{L^2} + \big((v.\nabla)v, v\big)_{L^2} + \big(\nabla p, v\big)_{L^2} = 0.$$

Remarquons alors que les deux derniers termes sont nuls puisque v est à divergence nulle ; en effet, on a $\big((v.\nabla)v, v\big)_{L^2} = 0$ par la propriété (1.1) et $\big(\nabla p, v\big)_{L^2} = -\big(p, \operatorname{div} v\big)_{L^2} = 0$. On a donc, par intégration par parties,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} = 0.$$

Enfin, on intègre en temps et on obtient l'égalité d'énergie suivante : pour tout $t \geq 0$, on a formellement

$$\frac{1}{2}\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(s)\|^2_{L^2}\,\mathrm{d}s = \frac{1}{2}\|v_0\|^2_{L^2}. \tag{1.4}$$

Cette égalité permet de se représenter l'effet régularisant de l'équation ; en effet, si la donnée initiale v_0 est dans $L^2(\mathbb{R}^n)^n$, alors la solution est dans $L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^n)^n)$. Par ailleurs, remarquons que, si v_0 est dans $L^2(\mathbb{R}^2)^2$, alors la solution v appartient à $L^\infty_T L^2(\mathbb{R}^2)^2 \cap L^2_T\dot{H}^1(\mathbb{R}^2)^2$. Par interpolation, on en déduit que $v \in L^3_T\dot{H}^{\frac{2}{3}}(\mathbb{R}^2)^2 = L^3_T\dot{H}^{\frac{2}{6}+\frac{1}{3}}(\mathbb{R}^2)^2$. Dans le cas de la dimension 2, l'égalité d'énergie (1.4) continue donc d'être vérifiée au cours du temps. Au contraire, en dimension 3, on constate que $\frac{2}{3} \neq \frac{5}{6}$, alors que nous avons supposé que v est dans $L^3_T\dot{H}^{\frac{5}{6}}(\mathbb{R}^3)^3$. Ceci explique que l'on ne demande que l'inégalité d'énergie dans la Définition 1.2.

L'égalité d'énergie (1.4) servira à définir les solutions de Leray et nous énoncerons brièvement des résultats d'existence globale de ce type de solutions. On considère le problème de Cauchy (1.2) avec v_0 une distribution donnée. On définit alors les solutions faibles des équations (NS) grâce à leur formulation variationnelle.

Définition 1.1 (Solution faible)**.** *Un champ de vecteurs* $v \in L^2_{loc}\big([0,T]\times\mathbb{R}^n\big)$ *est une solution faible de* (1.2) *si, pour tout champ de vecteur* $\varphi \in \mathcal{C}^\infty\big([0,T]\times\mathbb{R}^n\big)$ *à divergence nulle et à support compact en espace et pour tout* $0 \leq t \leq T$, *on a*

$$\int_{\mathbb{R}^n} v.\varphi(t,x)\,dx = \int_{\mathbb{R}^n} v_0(x).\varphi(0,x)\,dx + \int_0^t\!\!\int_{\mathbb{R}^n} \big(v.\Delta\varphi + v\otimes v : \nabla\varphi + v.\partial_t\varphi\big)(s,x)\,dx\,ds,$$

où le produit matriciel $v \otimes v : \nabla\varphi$ est défini par

$$v \otimes v : \nabla\varphi = \sum_{i,j=1}^{n} v_i v_j \partial_i \varphi_j.$$

Pour tenir compte de la conservation de l'énergie du sytème, Leray (voir [30]) introduit ce que l'on appelle les *solutions de Leray.*
Prenons $v_0 \in L^2(\mathbb{R}^n)^n$ comme donnée initiale pour le problème de Cauchy (1.2).

Définition 1.2 (Solution de Leray)**.** *Un champ de vecteurs $v \in L^\infty_T L^2(\mathbb{R}^n)^n \cap L^2_T H^1(\mathbb{R}^n)^n$ est une solution de Leray de* (1.2) *si v est une solution faible vérifiant, pour tout $0 \leq t \leq T$, l'inégalité d'énergie*

$$\frac{1}{2}\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(s)\|^2_{L^2}\, ds \leq \frac{1}{2}\|v_0\|^2_{L^2}.$$

Le premier résultat d'existence globale de solutions de Leray de (NS) a été obtenu par Leray lui-même [30] en 1934. En dimension 2, on a même un résultat d'unicité. Le théorème de Leray est le suivant :

Théorème 1.1 (Leray '34)**.** *Soit $v_0 \in L^2(\mathbb{R}^n)^n$ un champ de vecteurs à divergence nulle. Il existe alors une solution de Leray v au problème de Cauchy* (1.2) *pour tout $T \geq 0$.*
De plus, si $n = 2$, cette solution v est unique.

La méthode de Leray consiste à approcher (NS) par ce que l'on appelle les équations de Navier-Stokes régularisées (*mollified Navier-Stokes equations*). Prenons une fonction test positive $\rho \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ de moyenne égale à 1 et définissons $\rho_\varepsilon = \frac{1}{\varepsilon^n}\rho\left(\frac{\cdot}{\varepsilon}\right)$. Alors les équations de Navier-Stokes régularisées sont :

$$(MNS^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t v - \Delta v &=& -\mathbb{P}\big(\,(\rho_\varepsilon * v).\nabla\,\big)v - \nabla p \\ \operatorname{div} v &=& 0 \end{array} \right. .$$

En dimension 3, en revanche, on n'a pas existence globale et unicité d'une solution de (NS) dans $L^\infty L^2(\mathbb{R}^3)^3 \cap L^2 \dot{H}^1(\mathbb{R}^3)^3$. Mais le théorème suivant, démontré par Fujita et Kato [17] en 1964, montre que, si l'on se place dans un espace invariant par la même transformation que le système de Navier-Stokes

$$\lambda \mapsto v_\lambda(t,x) := \lambda v(\lambda^2 t, \lambda x),$$

alors on peut obtenir des résultats d'existence globale et d'unicité si l'on suppose la donnée initiale v_0 petite et d'existence locale et d'unicité sans restriction sur la taille de v_0. L'espace qui convient en dimension 3 est l'espace $L^\infty \dot{H}^{\frac{1}{2}} \cap L^2 \dot{H}^{\frac{3}{2}}$.

Théorème 1.2 (Fujita-Kato '64)**.** *Soit* $v_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ *à divergence nulle. Il existe un unique temps maximal* T^* *et une unique solution locale* $v \in L^\infty_{loc} \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3 \cap L^2_{loc} \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)^3$.
De plus, il existe une constante $C > 0$ *telle que, pour toute donnée initiale* v_0 *vérifiant*

$$\|v_0\|_{\dot{H}^{\frac{1}{2}}} \leq C,$$

la solution v *est globale,* i.e. $T^* = +\infty$.

La méthode de Fujita et Kato est différente de celle de Leray ; en effet, ils utilisent le formalisme des solutions *mild*, c'est-à-dire qu'ils considèrent l'équation de Navier-Stokes comme une équation de la chaleur et utilisent le semi-groupe de la chaleur pour montrer qu'il existe une unique solution v de (NS) qui est point fixe de l'application ϕ définie par :

$$\phi(v)(t,x) = e^{t\Delta} v_0(x) + \int_0^t e^{(t-s)\Delta} \mathbb{P}(v.\nabla)v(s,x)\,\mathrm{d}s.$$

Les théorèmes de Leray et de Fujita et Kato seront très utiles dans les chapitres 2 et 3. Nous approcherons, de deux façons différentes, l'unique solution forte de (NS) donnée par le théorème de Leray en dimension 2 et celle donnée par le théorème de Fujita et Kato en dimension 3.

1.2 Motivations de la thèse

Les équations de Navier-Stokes sont, encore aujourd'hui, assez peu comprises, surtout en dimension 3. Le but est donc d'essayer de mieux comprendre le comportement qualitatif, mais aussi quantitatif (via des simulations numériques), des solutions lorsque l'on sait qu'elles existent. Pour notre part, dans cette thèse, nous nous concentrons sur le comportement qualitatif des solutions.

Dans cette optique, nous approchons les solutions de (NS) par des solutions d'équations hyperboliques mieux connues. Nous considérerons notamment une équation d'ondes amorties, dont les solutions se propageront instantanément, et une équation d'ondes faiblement compressibles qui aura une vitesse de propagation finie.

Dans un premier temps, nous nous sommes intéressés à une approximation due à Cattaneo en 1949 pour l'étude de l'équation de la chaleur (voir [9,10]) dans un corps au repos

$$\partial_t \theta - \Delta\theta = 0.$$

Cette équation viole le principe de la relativité à cause de la propagation instantanée de ses solutions. Pour remédier à ce problème, Cattaneo [9,10] et d'autres (Chester, Vernotte, etc.) ont proposé le modèle hyperbolique suivant :

$$\frac{1}{C^2}\partial_{tt}\theta + \frac{1}{\alpha}\partial_t\theta - \Delta\theta = 0.$$

Cette équation s'appelle *l'équation du Télégraphe*. Elle a une vitesse de propagation finie et est compatible avec le principe de relativité et avec la deuxième loi de la thermodynamique ; elle constitue donc un modèle physique satisfaisant.

Cette perturbation, vue comme une relaxation des équations d'Euler, a été considérée plus tard par Brenier, Natalini et Puel dans [7] puis par Paicu et Raugel dans [36] (une variante de cette approximation a aussi été étudiée par Natalini et Rousset dans [34]). De même que celle de Cattaneo, la perturbation que l'on étudie ici consiste à rajouter le terme $\varepsilon\partial_{tt}v$ à la première équation dans (NS), donnant ainsi l'équation des ondes amorties suivante :

$$(HNS^\varepsilon) \quad \left\{ \begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon & = & -\mathbb{P}(u^\varepsilon.\nabla)u^\varepsilon \\ \operatorname{div} u^\varepsilon & = & 0 \end{array} \right. .$$

Notons que, vu le caractère non-local du projecteur de Leray $\mathbb{P}$, le système (HNS^ε) a une vitesse de propagation infinie.

Ce modèle a été justifié dans plusieurs articles par différentes méthodes. Citons par exemple [22] où les auteurs considèrent ces équations comme un modèle de formation de glace dans les lacs. Du point de vue de l'analyse numérique, citons [25,26] où l'approximation (HNS^ε) est utilisée pour calculer les solutions de (NS) et est justifiée par relaxation, comme dans [7].

Nous verrons dans la sous-section 1.3.1 comment Brenier, Natalini et Puel justifient cette approximation à travers une relaxation diffusive des équations d'Euler et quels résultats ils ont obtenu dans [7]. Dans la sous-section 1.3.2, nous donnerons les résultats de Paicu et Raugel dans [35,36] en expliquant la différence entre leur méthode et celle de [7]. Nous avons obtenu des résultats d'existence globale et de convergence qui prolongent et complètent ceux de [7] et [35, 36]. Nous les énoncerons brièvement dans la sous-section 1.4.1 et les détaillerons dans le chapitre 2.

L'idée centrale de ce travail est d'approcher les solutions de (NS) par les solutions d'une nouvelle perturbation hyperbolique à vitesse de propagation finie, ce qui permet en même temps de mieux comprendre le comportement qualitatif des solutions de (NS) et d'envisager des simulations numériques en utilisant ce nouveau modèle.

C'est ce que nous avons fait dans l'article [21] en considérant l'équation

$$(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla \operatorname{div} u^{\varepsilon,\alpha}$$

qui est inspirée, d'une part, par l'approximation de Cattaneo et, d'autre part, par un modèle considéré par Višik et Fursikov dans [40] pour l'étude de solutions statistiques de (NS). Parmi tous les travaux utilisant ce dernier modèle, citons [2, 16, 27]. Dans la sous-section 1.4.2, nous introduirons l'équation $(HNS^{\varepsilon,\alpha})$, nous énoncerons brièvement les résultats obtenus sur cette équation et nous donnerons certaines propriétés de ses solutions. Le chapitre 3 sera consacré à l'étude détaillée de cette équation.

1.3 Travaux précédents

1.3.1 Brenier, Natalini et Puel

Dans leur article [7], Brenier, Natalini et Puel ont considéré le système (HNS^{ε}) sur $\mathbb{R}^+ \times \mathbb{T}^2$, où $\mathbb{T}^2$ est le tore bi-dimensionnel $\mathbb{R}^2/\mathbb{Z}^2$. Ils ont démontré la convergence des solutions globales de (HNS^{ε}) avec données initiales dans $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^2$ vers les solutions de (NS) avec une donnée régulière.

Tout d'abord, les auteurs remarquent que le système qu'ils étudient intervient aussi dans un modèle de relaxation des équations d'Euler incompressibles

$$(\mathcal{E}) \quad \begin{cases} \partial_\tau u &= -(u.\nabla)u - \nabla p \\ \operatorname{div} u &= 0 \\ u|_{\tau=0} &= u_0 \end{cases}$$

à la Jin-Xin. Cette relaxation est la suivante (voir [7, 23]) :

$$\left\{\begin{array}{rcl} \partial_\tau u + \operatorname{div} V &=& \nabla q \\ \partial_\tau V + \nabla u &=& -\frac{1}{\eta}(V - u\otimes u) \\ \operatorname{div} u &=& 0 \\ (u,V)|_{\tau=0} &=& (u_0, V_0) \end{array}\right. .$$

Remarquons que, lorsque η tend vers 0, on retrouve formellement le système $(\mathcal{E})$. Considérons maintenant le changement d'échelle diffusif

$$u^\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}} u(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}) \ ,$$

$$V^\varepsilon(t,x) = \frac{1}{\varepsilon} V(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}) \ , \quad q^\varepsilon(t,x) = \frac{1}{\varepsilon} q(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}})$$

et posons $\eta = 1$. On obtient alors le système d'équations

$$(S^\varepsilon) \quad \left\{\begin{array}{rcl} \partial_t u^\varepsilon + \operatorname{div} V^\varepsilon + \nabla p^\varepsilon &=& 0 \\ \partial_t V^\varepsilon + \frac{1}{\varepsilon}\nabla u^\varepsilon &=& \frac{1}{\varepsilon}(u^\varepsilon \otimes u^\varepsilon - V^\varepsilon) \\ \operatorname{div} u^\varepsilon &=& 0 \\ (u^\varepsilon, V^\varepsilon)|_{t=0} &=& (u_0^\varepsilon, V_0^\varepsilon) \end{array}\right.$$

que l'on peut réécrire sous la forme d'une équation d'ondes amorties :

$$(HNS^\varepsilon) \quad \left\{\begin{array}{rcl} \varepsilon\partial_{tt} u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &=& -\mathbb{P}(u^\varepsilon.\nabla)u^\varepsilon \\ \operatorname{div} u^\varepsilon &=& 0 \\ (u^\varepsilon, V^\varepsilon)|_{t=0} &=& (u_0^\varepsilon, V_0^\varepsilon) \end{array}\right. .$$

Formellement, le système (S^ε) converge vers (NS) lorsque ε tend vers 0. Dans [7], les auteurs montrent que cette convergence n'est pas seulement formelle. Le résultat est le suivant :

Théorème 1.3 (Brenier-Natalini-Puel '04)**.** *Soit v_0 un champ de vecteurs régulier à divergence nulle sur $\mathbb{T}^2$. Soit $(u_0^\varepsilon, V_0^\varepsilon)$ une suite de données initiales pour le système relaxé (S^ε). Supposons qu'il existe une constante C telle que*

$$\|u_0^\varepsilon - v_0\|_{L^2}^2 \leq C\sqrt{\varepsilon} \ , \quad \|u_0^\varepsilon\|_{H^1} + \|\partial_t u^\varepsilon(0,.)\|_{L^2} \leq C \ ,$$

$$\|u_0^\varepsilon\|_{L^2} + \|curl\, u_0^\varepsilon\|_{L^2} + \|\nabla(curl\, u_0^\varepsilon)\|_{L^2} < \frac{C_0}{K_s\sqrt{\varepsilon}} \ ,$$

où $C_0 < 1$ et K_s est la constante intervenant dans l'inégalité de Sobolev sur $H^2(\mathbb{T}^2)$.
Alors, si v est la solution régulière de (NS) avec donnée initiale v_0, pour tout $T \geq 0$, il existe une constante C_T telle que

$$\sup_{t\in[0,T]} \|u^\varepsilon(t) - v(t)\|_{L^2}^2 \leq C_T\sqrt{\varepsilon}.$$

Pour démontrer ce résultat, les auteurs utilisent une méthode d'énergie hyperbolique et l'énergie modulée de Dafermos pour la preuve de la convergence. L'énergie associée à (HNS^ε) est (dans le tore $\mathbb{T}^2$)

$$E_\varepsilon(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2}|u^\varepsilon(t) + \varepsilon\partial_t u^\varepsilon(t)|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon(t)|^2 + \varepsilon|\nabla u^\varepsilon(t)|^2 \right) \mathrm{d}x.$$

Elle est obtenue en multipliant l'équation par $u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon$ puis en intégrant en espace. En *modulant* cette énergie par une solution v de (NS), on obtient l'énergie modulée de Dafermos

$$E_{\varepsilon,v}(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2}|u^\varepsilon(t) - v(t) + \varepsilon\partial_t u^\varepsilon(t)|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon(t)|^2 + \varepsilon|\nabla u^\varepsilon(t)|^2 \right) \mathrm{d}x$$

qui contrôle la norme $\|u^\varepsilon(t) - v(t)\|_{L^2}^2$.

Cette énergie modulée a été utilisée dans plusieurs travaux sur l'approximation des équations paraboliques par des équations hyperboliques. Citons par exemple l'article [5] de Brenier où il montre la convergence du système de Vlasov-Poisson dans un régime quasi-neutre vers le système d'Euler incompressible, en adaptant la méthode utilisée par Lions [31] pour établir la convergence des solutions de Leray de (NS) en dimension 3 vers des solutions dissipatives des équations d'Euler.

Signalons aussi qu'à plusieurs reprises dans [7], les auteurs utilisent l'estimation suivante sur la solution u^ε de (HNS^ε) :

$$\|u^\varepsilon\|_{L^\infty\left(\mathbb{R}^+;L^\infty(\mathbb{T}^2)^2\right)} = \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Nous verrons plus tard que nous avons aussi besoin de cette estimation et nous la montrerons. Signalons enfin que la méthode utilisée par Brenier-Natalini-Puel est restreinte au cadre bi-dimensionnel et demande beaucoup de régularité sur les données initiales.

1.3.2 Paicu et Raugel

Dans leurs articles [35, 36], Paicu et Raugel approchent l'équation de Navier-Stokes avec une force extérieure f pouvant dépendre du temps

$$(NS) \quad \left\{ \begin{array}{rcl} \partial_t v - \Delta v & = & -\mathbb{P}(v.\nabla)v + f \\ \operatorname{div} v & = & 0 \end{array} \right.$$

par la perturbation hyperbolique (HNS^ε). Grâce à l'utilisation conjointe d'une méthode d'énergie similaire à celle de [7] et d'une inégalité de Strichartz sur les hautes fréquences (démontrée en annexe A), ils obtiennent des résultats d'existence globale et d'unicité pour l'équation d'ondes (HNS^ε) avec données initiales dans $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ ou dans $H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$. De plus, ils obtiennent des résultats de convergence sur les solutions $(u^\varepsilon, \partial_t u^\varepsilon)$ de (HNS^ε) qui améliorent significativement ceux de Brenier, Natalini et Puel.

Résultats 2D

Dans [35], Paicu et Raugel montrent des résultats d'existence globale et d'unicité pour (HNS^ε) dans $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ (et même dans des espaces un peu moins réguliers), en utilisant une estimation de Strichartz sur les hautes fréquences.

Grâce à l'utilisation conjointe des estimations d'énergie (dans [7]) et de l'estimation de Strichartz sur l'équation des ondes amorties (HNS^ε), Paicu et Raugel améliorent significativement les résultats de [7] dans deux directions :

- Ils considèrent des données initiales pour (HNS^ε) beaucoup moins régulières. Rappelons que les données initiales sont supposées être dans $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^2$ dans [7], alors qu'elles peuvent être seulement dans $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ dans [35, 36].

- Ils comparent la solution $(u^\varepsilon, \partial_t u^\varepsilon)$ de (HNS^ε) à la solution $(v^*, \partial_t v^*)$ du problème de Cauchy

$$\left\{ \begin{array}{rcl} \partial_t v^* - \Delta v^* & = & -\mathbb{P}(v^*.\nabla)v^* + \mathbb{P}f^\varepsilon \\ \operatorname{div} v^* & = & 0 \\ v^*|_{t=0} & = & u_0^\varepsilon \end{array} \right.$$

  en norme $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$.

Plus précisément, ils obtiennent les résultats suivants que nous énonçons ici sans force extérieure, pour ne pas alourdir les notations.

Théorème 1.4 (Paicu-Raugel '07 - Existence globale)**.** *Soit* $R > 0$. *Il existe un nombre positif* ε_1, *ne dépendant que de* R, *tel que, pour tout* $0 < \varepsilon < \varepsilon_1$, *si le couple de vecteurs à divergence nulle* $(u_0^\varepsilon, u_1^\varepsilon) \in H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ *vérifie*

$$\|u_0^\varepsilon\|_{L^2} + \|u_1^\varepsilon\|_{L^2} + \|u_0^\varepsilon\|_{\dot{H}^1} \leq R,$$

alors l'équation (HNS^ε) *admet une unique solution intégrale globale* $(u^\varepsilon, \partial_t u^\varepsilon)$ *dans* $C^0\big(\mathbb{R}^+, H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2\big)$.

Quand ε tend vers 0, la limite formelle de (HNS^ε) est (NS). Dans le théorème suivant, Paicu et Raugel montrent que cette limite n'est pas seulement formelle.

Théorème 1.5 (Paicu-Raugel '07 - Convergence)**.** *Soit* $\beta > 0$. *Soient* R *et* T *deux nombres strictement positifs. Il existe* $\varepsilon_1 > 0$, *ne dépendant que de* R *et* T, *tel que, pour tout* $0 < \varepsilon < \varepsilon_1$, *si le couple de vecteurs à divergence nulle* $(u_0^\varepsilon, u_1^\varepsilon) \in H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ *vérifie*

$$\|u_0^\varepsilon\|_{H^1} + \sqrt{\varepsilon}\|u_1^\varepsilon\|_{L^2} \leq R,$$

alors l'équation (HNS^ε) *admet une unique solution intégrale*

$$(u^\varepsilon, \partial_t u^\varepsilon) \in C^0\big(\mathbb{R}^+, H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2\big)$$

et on a, pour tout $0 \leq t \leq T$,

$$\sqrt{\varepsilon}\left\|\partial_t\Big(t\big(u^\varepsilon(t) - v^*(t)\big)\Big)\right\|_{L^2} + \big\|t\big(u^\varepsilon(t) - v^*(t)\big)\big\|_{H^1} \leq \varepsilon e^K,$$

où K *est une constante ne dépendant que de* R *et* T.

Résultats 3D

En dimension 3, Paicu et Raugel obtiennent des résultats similaires à ceux de la dimension 2. Sous des hypothèses de régularité et de petitesse sur les données initiales légèrement plus restrictives que dans le Théorème 1.4, ils montrent qu'il existe des solutions globales de (HNS^ε) avec données initiales dans $H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$, où $\delta \in \big(0, \frac{1}{2}\big)$.

Théorème 1.6 (Paicu-Raugel '07 - Existence globale)**.** *Pour tout* $0 < \delta < \frac{1}{2}$, *il existe une constante* $K = K(\delta)$ *telle que, pour tout* $R > 0$, *il existe un nombre réel strictement positif* $\varepsilon_1 = \varepsilon_1(R)$ *tel que, pour tout* $0 < \varepsilon \leq \varepsilon_1$ *et pour tout couple de vecteurs à divergence nulle* $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$ *vérifiant*

$$\|u_0^\varepsilon\|_{L^2} \times \|\nabla u_0^\varepsilon\|_{L^2} \leq K$$

et

$$\|u_0^\varepsilon\|_{L^2} + \|\nabla u_0^\varepsilon\|_{H^\delta} + \|u_1^\varepsilon\|_{H^\delta} \leq R,$$

l'équation (NLW^ε) *admet une unique solution intégrale globale*

$$(u^\varepsilon, \partial_t u^\varepsilon) \in \mathcal{C}^0(\mathbb{R}^+, H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3).$$

De plus, un résultat de convergence analogue à celui de la dimension 2 peut être obtenu en dimension 3.

1.3.3 Autres approximations de (NS)

Dans cette partie, nous allons présenter brièvement les travaux de Natalini et Rousset [34], Carfora et Natalini [8] et Jobic, Natalini et Pavan [24] qui portent sur des approximations hyperboliques des équations de Navier-Stokes incompressibles différentes de (HNS^ε) et $(HNS^{\varepsilon,\alpha})$.

Dans [34], Natalini et Rousset ont considéré une variante de la perturbation (HNS^ε) qui approche formellement les équations de Navier-Stokes incompressibles. Leur approximation

$$\left\{\begin{array}{rcl} \partial_t u^\varepsilon + \operatorname{div} V^\varepsilon &=& \nabla\varphi^\varepsilon \\ \varepsilon\partial_t V^\varepsilon + a^2\nabla u^\varepsilon &=& u^\varepsilon\otimes u^\varepsilon - V^\varepsilon \\ \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) &=& 0 \\ \Delta\varphi^\varepsilon &=& \frac{\rho^\varepsilon-1}{\varepsilon} \end{array}\right.$$

combine la méthode de relaxation des équations d'Euler avec changement d'échelle diffusif (utilisée dans [7]) et la limite quasi-neutre des équations d'Euler-Poisson, c'est-à-dire où l'on fait tendre la viscosité vers 0. Dans leur papier, les auteurs justifient rigoureusement cette approximation en utilisant une méthode d'énergie hyperbolique.

Par ailleurs, dans [4, 8, 24], les auteurs considèrent des approximations des équations de Navier-Stokes combinant une approche cinétique et une relaxation. D'abord, appliquant le changement d'échelle parabolique $(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ à la solution $f(t,x,v)$ de l'équation de Boltzmann, on obtient

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v.\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} Q(f^\varepsilon, f^\varepsilon)\,.$$

Puis on utilise l'approximation BGK (ou opérateur de Bhatnagar–Gross–Krook), consistant en un modèle particulier pour l'opérateur de collisions Q. On se ramène alors au problème suivant : trouver $f_i^\varepsilon \in \mathbb{R}^{d+1}$ vérifiant

$$\left\{\begin{array}{l} \partial_t f_i^\varepsilon + \frac{1}{\varepsilon}\lambda_i.\nabla_x f_i^\varepsilon = \frac{1}{\tau\varepsilon^2}\Big(M_i(\rho^\varepsilon, \varepsilon\rho u^\varepsilon) - f_i^\varepsilon\Big) \\ f_i^\varepsilon(0,x) = M_i(\bar\rho, \varepsilon\bar\rho u_0)\,,\ i = 1,\cdots,N \\ \rho^\varepsilon := \sum_{i=1}^N f_{i,0}^\varepsilon\,,\ \varepsilon\rho u_l^\varepsilon := \sum_{i=1}^N f_{i,l}^\varepsilon \end{array}\right. ,$$

où M_i (appelée Maxwellienne) vérifie $Q(M_i, M_i)$.

Dans [24], les auteurs considèrent ce modèle à vitesse de propagation finie et développent un schéma permettant d'étudier numériquement la limite de l'inconnue u^ε de ce système vers une solution des équations de Navier-Stokes incompressibles.

1.4 Résultats de la thèse

1.4.1 L'équation des ondes amorties

Pour commencer, rappelons l'approximation considérée dans cette partie. On rajoute le terme $\varepsilon\partial_{tt}v$ à la première équation de (NS) et on obtient le système suivant :

$$(HNS^\varepsilon) \quad \left\{ \begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &=& -(u^\varepsilon.\nabla)u^\varepsilon - \nabla p^\varepsilon \\ \operatorname{div} u^\varepsilon &=& 0 \end{array} \right. .$$

De même que plus haut, pour les équations de Navier-Stokes, on peut exprimer l'inconnue p^ε en fonction de u^ε en appliquant l'opérateur div à la première équation de (HNS^ε). Le système se réécrit alors

$$(HNS^\varepsilon) \quad \left\{ \begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &=& -\mathbb{P}(u^\varepsilon.\nabla)u^\varepsilon \\ \operatorname{div} u^\varepsilon &=& 0 \end{array} \right. .$$

L'équation obtenue est une équation d'ondes non-linéaire avec amortissement, dû au terme $+\partial_t u^\varepsilon$ qui, grâce au signe $+$, dissipe de l'énergie. Par abus de notation, nous confondrons souvent les systèmes (HNS^ε) et (NS) avec leur première équation.

Dans le chapitre 2, nous montrerons que la solution de (HNS^ε), avec des données initiales bien choisies, approche l'unique solution forte de (NS), donnée par le Théorème 1.1 ou 1.2 (selon la dimension), avec une donnée initiale v_0 de régularité quasi-optimale : $v_0 \in \dot{H}^{0^+}(\mathbb{R}^2)^2$ ou $v_0 \in \dot{H}^{\left(\frac{1}{2}\right)^+}(\mathbb{R}^3)^3$. Plus précisément, nous obtenons le résultat suivant en dimension 2 :

Théorème 1.7 (Le cas 2D)**.** *Soient* $0 < s, \delta < 1$. *Soit* $v_0 \in H^s(\mathbb{R}^2)^2$ *un champ de vecteurs à divergence nulle et soit* $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ *une suite de données initiales à divergence nulle du système* (HNS^ε). *Supposons que les conditions suivantes soient vérifiées :*

$$\left\{ \begin{array}{rcl} \|u_0^\varepsilon - v_0\|_{L^2} + \varepsilon\|u_1^\varepsilon\|_{L^2} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^1} &=& \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{1+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^\delta} &=& \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^\delta} &=& o(1)\,. \end{array} \right.$$

Alors, si ε *est assez petit, l'équation* (HNS^ε) *admet une unique solution globale* u^ε *qui converge, en norme* $L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)^2)$, *lorsque* ε *tend vers* 0, *vers l'unique solution forte* v *de* (NS), *avec donnée initiale* v_0. *De plus, pour tout* $T > 0$, *il existe une constante* C_T, *qui ne dépend que de* T *et* v, *telle que*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} |u^\varepsilon - v|^2\, dx \le C_T \varepsilon^{\left(\frac{s}{2}\right)^-}.$$

En dimension 3, en imposant une condition de petitesse supplémentaire (mais naturelle, d'après le Théorème 1.2) sur la donnée initiale u_0^ε, ou de façon équivalente sur v_0, nous obtenons un résultat analogue à celui de la dimension 2 :

Théorème 1.8 (Le cas 3D). *Soient* $0 < s, \delta < 1$. *Soient* $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ *un champ de vecteurs à divergence nulle et* $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ *une suite de données initiales à divergence nulle pour* (HNS^ε) *telle que* $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$. *Supposons que les conditions suivantes soient vérifiées :*

$$\left\{\begin{array}{rcl} \|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} & = & o(1)\,. \end{array}\right.$$

Alors, si ε *est assez petit, l'équation* (HNS^ε) *admet une unique solution intégrale globale* u^ε *qui converge, en norme* $L^\infty_{loc}(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3)$, *lorsque* ε *tend vers* 0, *vers l' unique solution* v *de* (NS), *avec donnée initiale* v_0. *De plus, pour tout* $T > 0$, *il existe une constante* C_T, *qui ne dépend que de* T *et* v, *telle que*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |\Lambda^{\frac{1}{2}}(u^\varepsilon - v)|^2\, dx \leq C_T \varepsilon^{\left(\frac{s}{2}\right)^-}.$$

Nous utilisons une méthode d'énergie hyperbolique inspirée de [7] et raffinée pour, d'une part, s'étendre à la dimension 3 et, d'autre part, requérir moins de régularité sur les données initiales. Nous nous fondons sur des estimations de produit et des inégalités d'interpolation pour obtenir des estimées assez précises sur le terme non-linéaire $(u^\varepsilon.\nabla)u^\varepsilon$.

Remarquons que les conditions de petitesse requises sur les données initiales sont moins restrictives que celles de [35, 36]. En revanche, vu que l'on n'utilise pas d'inégalité de Strichartz, nous avons besoin de plus de régularité pour l'existence globale. Par ailleurs, nous avons un résultat de convergence uniquement $L^\infty_{\text{loc}}\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n$ mais pour des données initiales de (NS) à régularité quasi-optimale.

Le chapitre 2 sera consacré à l'étude de la limite de l'équation d'ondes amorties (HNS^ε).

1.4.2 Le modèle faiblement compressible

Dans cette thèse, nous introduisons une nouvelle approximation hyperbolique de (NS), avec une vitesse de propagation finie, inspirée de la perturbation de Cattaneo [9,10] et d'un modèle faiblement compressible étudié par Višik et Fursikov [40]. Partant de notre première perturbation

$$(HNS^\varepsilon) \quad \left\{ \begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon & = & -(u^\varepsilon.\nabla)u^\varepsilon - \nabla p^\varepsilon \\ \operatorname{div} u^\varepsilon & = & 0 \end{array} \right. ,$$

nous pénalisons la *contrainte* $\operatorname{div} u^\varepsilon = 0$ de la façon suivante. Pour ne pas alourdir les notations, plaçons-nous dans $\mathbb{R}^2$. En appliquant l'opérateur div à l'équation, on obtient

$$0 = \operatorname{div} f(u^\varepsilon) - \Delta p^\varepsilon,$$

où $f(u^\varepsilon) = -(u^\varepsilon.\nabla)u^\varepsilon$. On considère alors l'équation stationnaire linéaire suivante :

$$-\Delta w^\varepsilon = f - \nabla p^\varepsilon \ , \quad \operatorname{div} w^\varepsilon = 0,$$

avec $f \in \dot{H}^{-1}$. La formulation variationnelle de cette équation est : pour toute fonction $v \in \dot{H}^1$ telle que $\operatorname{div} v = 0$,

$$\int_{\mathbb{R}^2} \left(-\Delta w^\varepsilon - f + \nabla p^\varepsilon\right).v \,\mathrm{d}x = 0. \tag{1.5}$$

Par intégration par parties, on obtient que $\int_{\mathbb{R}^2} \nabla p^\varepsilon.v = 0$ car $\operatorname{div} v = 0$ et la formulation variationnelle (1.5) devient

$$\int_{\mathbb{R}^2} \left(\nabla w^\varepsilon.\nabla v - f.v\right) \,\mathrm{d}x = 0.$$

On en déduit que w^ε minimise la fonctionnelle $J^\varepsilon(v) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla v|^2 - f.v\right) \mathrm{d}x$ dans l'espace $\dot{H}^1_\sigma(\mathbb{R}^2) = \left\{v \in \dot{H}^1(\mathbb{R}^2) : \operatorname{div} v = 0\right\}$.

On pénalise alors la contrainte $\operatorname{div} v = 0$ en minimisant non plus la fonctionnelle J^ε mais une nouvelle fonctionnelle

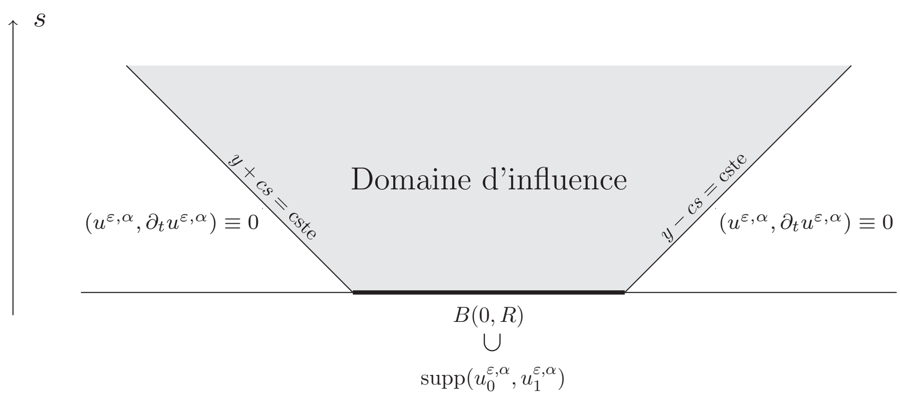
$$J^{\varepsilon,\alpha}(v) = J^\varepsilon(v) + \frac{1}{2\alpha}\int_{\mathbb{R}^2} |\operatorname{div} v|^2 \,\mathrm{d}x$$

dans l'espace $\dot{H}^1(\mathbb{R}^2)$. Supposons alors que $w^{\varepsilon,\alpha}$ soit un minimiseur de $J^{\varepsilon,\alpha}$. Formellement, on voit que, lorsque α tend vers 0, $\|\operatorname{div} w^{\varepsilon,\alpha}\|_{L^2}$ doit aussi tendre vers 0 sinon on aurait $J^{\varepsilon,\alpha}(w^{\varepsilon,\alpha}) \to +\infty$. Nous justifierons ce phénomène dans le chapitre 3.

En pénalisant ainsi la contrainte d'incompressibilité, on obtient l'équation hyperbolique suivante :

$$(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla\operatorname{div} u^{\varepsilon,\alpha}.$$

Nous montrerons dans le chapitre 3 que cette équation a une vitesse de propagation finie, c'est-à-dire qu'il existe un nombre positif $c = c(\varepsilon, \alpha)$ tel que, si l'on prend un couple de données initiales $(u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha})$ à support dans une boule $B(0, R)$ donnée, la solution $\big(u^{\varepsilon,\alpha}(t), \partial_t u^{\varepsilon,\alpha}(t)\big)$ sera nulle en dehors d'un certain cône que l'on appelle *domaine d'influence*, qui dépend de c.



Nous montrerons aussi que les solutions de $(HNS^{\varepsilon,\alpha})$ approchent celles de (NS) sous certaines conditions sur les données initiales. Pour cela, nous montrons que la solution $u^{\varepsilon,\alpha}$ de $(HNS^{\varepsilon,\alpha})$ avec données initiales $(u_0^{\varepsilon}, u_1^{\varepsilon})$ converge vers la solution u^{ε} de (HNS^{ε}) avec les mêmes données initiales puis nous utilisons les Théorèmes 1.7 et 1.8 pour conclure.

Théorème 1.9. *Soient $n = 2$ ou 3 et $0 < s, \delta < 1$. Soient $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ un champ de vecteurs à divergence nulle et*

$$(u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha}) = (u_0^{\varepsilon}, u_1^{\varepsilon}) \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$$

une suite de données initiales, indépendantes de α, à divergence nulle pour $(HNS^{\varepsilon,\alpha})$. Supposons que les conditions suivantes soient vérifiées :

$$\left\{ \begin{array}{rcl} \|u_0^{\varepsilon,\alpha} - v_0\|_{\dot H^{\frac{n}{2}-1}} + \varepsilon \|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1}} + \varepsilon^{\frac{1}{2}} \|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}} \|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}} \|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1+\delta}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}} \|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1+\delta}} & = & o(1) . \end{array} \right.$$

De plus, si $n = 3$, supposons que $\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)^3} < \frac{1}{36K_2^3}$, où K_2 est une constante telle que

$$\|f\|_{L^3(\mathbb{R}^3)} \leq K_2 \|f\|_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Alors, si ε et α sont assez petits, l'équation $(HNS^{\varepsilon,\alpha})$ admet des solutions globales qui approchent celles de (NS) en norme $L^\infty_{loc}\dot H^{\frac{n}{2}-1}(\mathbb{R}^n)^n$.

L'étude détaillée de l'approximation $(HNS^{\varepsilon,\alpha})$ fera l'objet du chapitre 3.

1.5 Plan de la thèse

Dans le chapitre 2, nous commençons par introduire dans la section 2.1 le modèle considéré ainsi que par un résumé des travaux précédents sur (HNS^ε). Nous prouverons le Théorème 1.7 dans la section 2.2 : nous montrerons d'abord l'existence globale de solutions dans $H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ dans la sous-section 2.2.1 en commençant par l'existence locale via le théorème de point fixe de Picard puis en globalisant les solutions locales obtenues. Nous utiliserons pour cela une méthode d'énergie hyperbolique. Enfin, nous prouverons que les solutions de (HNS^ε) approchent celles de (NS) lorsque ε tend vers 0.

Nous suivrons le même déroulement pour démontrer le Théorème 1.8. Nous montrons l'existence globale pour (HNS^ε) dans $H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ dans la section 2.3.1 en utilisant la même méthode qu'en dimension 2. Pour finir, on montre la convergence des solutions de (HNS^ε) vers celles des équations de Navier-Stokes quand ε tend vers 0.

Dans le chapitre 3, nous détaillerons l'étude de l'approximation $(HNS^{\varepsilon,\alpha})$. Dans la section 3.2, nous justifierons le choix de ce modèle puis nous montrerons dans la section 3.3 que ses solutions ont une vitesse de propagation finie. Ensuite, nous montrerons dans la section 3.4 qu'il existe des solutions locales $u^{\varepsilon,\alpha}$ pour $(HNS^{\varepsilon,\alpha})$.

La section 3.5 sera consacrée à l'étude du cas de la dimension 2 : nous rappelons d'abord des estimations du chapitre 2 dans la sous-section 3.5.1 puis, dans la sous-section 3.5.2, nous démontrerons que les solutions locales $u^{\varepsilon,\alpha}$ sont en fait globales en utilisant une méthode d'énergie hyperbolique. La convergence des solutions $u^{\varepsilon,\alpha}$ de $(HNS^{\varepsilon,\alpha})$ vers une solution u^ε de (HNS^ε) quand α tend vers 0 sera démontrée dans la sous-section 3.5.3.

De même, on démontre le théorème en dimension 3 dans la section 3.6 en commençant par quelques rappels du chapitre 2 dans la sous-section 3.6.1 puis en montrant l'existence globale dans 3.6.2 et, enfin, nous démontrons dans 3.6.3 la convergence de $u^{\varepsilon,\alpha}$ vers u^ε lorsque α tend vers 0.

Dans l'annexe A, nous démontrons l'inégalité de Strichartz sur l'équation d'ondes amorties

$$\partial_{tt}u + \partial_t u - \Delta u = G$$

utilisée par Paicu et Raugel dans [36]. Rappelons que, grâce à cette estimation de Strichartz, les auteurs obtiennent des hypothèses de régularité minimales pour les données initiales.

Enfin, nous rappelons quelques propriétés importantes de la théorie dyadique de Littlewood-Paley qui interviennent de façon cruciale dans les estimations du chapitre 3.

Chapitre 2

A damped wave equation as a perturbation of (NS)

In two and three space dimensions, and under suitable assumptions on the initial data, we show global existence for a damped wave equation which approaches, in some sense, the Navier-Stokes problem. The proofs are based on a refinement of the energy method in [7].

In this chapter, we improve the results of [7] and [36]. We relax the regularity of the initial data of the former, even though we still use energy methods as a principal tool. Regarding [36], the improvement consists in the simplicity of the proofs due to the non-use of Strichartz estimates and in requiring less restrictions on the data size, particularly in the 3D case. The convergence result we obtain is near-optimal regularity.

2.1 Introduction

First, let us recall the Navier-Stokes equations that describe the motion of a viscous, homegeneous and incompressible newtonian fluid

$$(NS) \quad \left\{ \begin{array}{rcl} \partial_t v - \Delta v &=& -\nabla : v \otimes v - \nabla p \\ \operatorname{div} v &=& 0 \\ v|_{t=0} &=& v_0 \end{array} \right. ,$$

where v is the velocity of the fluid and p the pressure, assumed to vanish at infinity, in the sense that

$$\frac{1}{|B(0,R)|} \int_{B(0,R)} p \, \mathrm{d}x \quad \longrightarrow \; 0 \; , \quad R \to +\infty.$$

The hyperbolic version of the Navier-Stokes equations studied here has been obtained after relaxation of the Euler equations and rescaling variables (see subsection 1.3.1 or [7] and references therein for the details) :

$$(HNS^\varepsilon) \quad \begin{cases} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &= -\nabla : u^\varepsilon \otimes u^\varepsilon - \nabla p^\varepsilon \\ \operatorname{div} u^\varepsilon &= 0 \\ (u^\varepsilon, \partial_t u^\varepsilon)|_{t=0} &= (u_0^\varepsilon, u_1^\varepsilon) \end{cases} .$$

This type of approximation is the same as the hyperbolic perturbation of the heat equation introduced and investigated by Cattaneo in the early fifties (see [9, 10] and further works).

In [7], Brenier, Natalini and Puel proved global existence and uniqueness for the perturbed Navier-Stokes equation (HNS^ε) with initial data in $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^2$, where $\mathbb{T}^2$ is the unit periodic square $\mathbb{R}^2/\mathbb{Z}^2$, if $\varepsilon < \varepsilon_0$ and ε_0 depends only on the initial data $(u_0^\varepsilon, u_1^\varepsilon)$. Moreover, they proved the convergence of the solution to (HNS^ε) towards a smooth solution to (NS) in the $L^\infty\big((0,T); L^2(\mathbb{R}^2)\big)$ norm, provided that v_0 is smooth.

In [36], Paicu and Raugel considered the same relaxed model (HNS^ε). In their paper, they proved, again if ε is small enough, global existence and uniqueness results for (HNS^ε) with significantly improved regularity for the initial data. In fact, they need only $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ regularity (and even less), due to the use of a Strichartz estimate. In the three-dimensional case, they state a global existence result under a somewhat restrictive smallness assumption on the initial data in $H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$, for $\delta > 0$ and ε small enough. Furthermore, they announced improved error estimates in the $H^1(\mathbb{R}^2)^2$ norm for $u^\varepsilon - v$ and in the $L^2(\mathbb{R}^2)^2$ norm for $\partial_t\left(t(u^\varepsilon - v)\right)$ in the two-dimensional case, where u^ε is as above and v is the solution to (NS) with $v_0 = u_0^\varepsilon \in H^1(\mathbb{R}^2)^2$ or $v_0 = u_0^\varepsilon \in H^{1+\delta}(\mathbb{R}^3)^3$.

In two and three space dimensions and under suitable smallness assumptions on the initial data, we prove global existence for the initial value problem (HNS^ε) in $\dot H^{\frac{n}{2}+\delta} \cap \dot H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ in dimensions $n = 2$ and $n = 3$. Moreover, for all positive T, we prove the convergence in the $L^\infty((0,T); \dot H^{\frac{n}{2}-1}(\mathbb{R}^n)^n)$ norm, with $n = 2, 3$, of solutions to (HNS^ε) towards solutions to the Navier-Stokes problem (NS) with initial data $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ and $s > 0$. More precisely, we prove the following two theorems.

Theorem 2.1 (2D case). *Let $0 < s, \delta < 1$. Let $v_0 \in H^s(\mathbb{R}^2)^2$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ be a sequence of initial data for problem (HNS^ε). Assume*

$$\left\{ \begin{array}{rcl} \|u_0^\varepsilon - v_0\|_{L^2} + \varepsilon\|u_1^\varepsilon\|_{L^2} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^1} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{1+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^\delta} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^\delta} & = & o\,(1)\,. \end{array} \right. \tag{2.1}$$

Then, for ε small enough, there exists a global solution u^ε to system (HNS^ε) that converges, when ε goes to 0, in the $L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)^2)$ norm, towards the unique solution v to the incompressible Navier-Stokes equations (NS), with v_0 as initial data. Moreover, for all positive T, there exists a constant C_T, depending only on T and v, such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} |u^\varepsilon - v|^2\, dx \leq C_T \varepsilon^{\left(\frac{s}{2}\right)^-}. \tag{2.2}$$

Theorem 2.2 (3D case). *Let $0 < s, \delta < 1$. Let $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ be a sequence of initial data for problem (HNS^ε) such that $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$. Assume*

$$\left\{ \begin{array}{rcl} \|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} & = & o\,(1)\,. \end{array} \right. \tag{2.3}$$

Then, for ε small enough, there exists a global solution u^ε to system (HNS^ε) that converges, when ε goes to 0, in the $L^\infty_{loc}(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3)$ norm, towards the unique solution v to the incompressible Navier-Stokes equations (NS), with v_0 as initial data. Moreover, for all positive T, there exists a constant C_T, depending only on T and v, such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |\Lambda^{\frac{1}{2}}(u^\varepsilon - v)|^2\, dx \leq C_T \varepsilon^{\left(\frac{s}{2}\right)^-}. \tag{2.4}$$

Remark. *As a consequence of the assumptions $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$ and $\|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we obtain the smallness of $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$, which is a necessary condition to the existence of global solutions to the Navier-Stokes equations in $\mathbb{R}^3$.*

Remark. *We prove the convergence for initial data $v_0 \in \dot{H}^{s+\frac{n}{2}-1}(\mathbb{R}^n)^n$, where $s > 0$ and $\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n$ is the critical space for the Navier-Stokes equations.*

Before going further, we should check that, given v_0, there exists a couple of functions $(u_0^\varepsilon, u_1^\varepsilon)$ satisfying all assumptions (2.1) in Theorem 2.1. First, since u_1^ε is not involved in the condition $\|u_0^\varepsilon - v_0\|_{L^2} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we can take $u_1^\varepsilon \equiv 0$. Then let u_0^ε be defined by

$$\widehat{u_0^\varepsilon}(\xi) = \widehat{v_0}(\xi)\ \mathbf{1}_{|\xi|<\frac{1}{\sqrt{\varepsilon}}}.$$

Now a Bernstein inequality (see [28] page 24) gives

$$\|u_0^\varepsilon\|_{\dot{H}^\sigma} \leq \left(\sqrt{\varepsilon}\right)^{s-\sigma} \|v_0\|_{\dot{H}^s}$$

and easy calculations lead to the Jackson inequality

$$\|u_0^\varepsilon - v_0\|_{L^2} \leq \left(\sqrt{\varepsilon}\right)^{s} \|v_0\|_{\dot{H}^s}.$$

Therefore, the conditions (2.1) are fulfilled by at least this particular choice of $(u_0^\varepsilon, u_1^\varepsilon)$. The same arguments lead to the existence of a couple of functions satisfying the assumptions (2.3) in Theorem 2.2.

The energy method we use in this paper is inspired by [7] and refined in order to take into account the loss of regularity of the initial data. Therefore, we frequently have to use product estimates and interpolations ; this makes the proofs more technical than in [7]. Moreover, we introduce a new energy, inspired from the classical one in [7, 36], which is more convenient for the spaces we work in. Let us notice that we do not use any Strichartz estimate in this paper so the proofs are easier to understand than those in [36]. Nevertheless, we lose the natural Strichartz regularity for the global existence results.

This chapter is organized as follows. In the next section, we treat the two-dimensional case and prove Theorem 2.1. In the last section, we adapt the results to the three-dimensional case and prove Theorem 2.2.

2.2 The 2D case : proof of Theorem 2.1

This section includes two subsections. In the first one, we prove global existence for the perturbation (HNS^ε) using a fixed point method. Then, in subsection 2.2.2, we prove the convergence of the global solutions u^ε towards a solution v to the Navier-Stokes equations, which is the last part of the statement of Theorem 2.1.

Notation : In the following, let $\mathbb{P}$ denote the Leray projection that maps a vector field to its zero-divergence part and, to alleviate the notations, let $L^p_T\mathcal{E}$ denote the space $L^p\left((0,T);\mathcal{E}\right)$.

2.2.1 Global existence in $\dot{H}^{1+\delta}(\mathbb{R}^2)\times\dot{H}^{\delta}(\mathbb{R}^2)$

Since $\mathbb{P}$ is a convolution operator on $\mathbb{R}^2$, it commutes with the differential operators. So, applying $\mathbb{P}$ to problem (HNS^ε), we obtain the damped (nonlinear) wave equation

$$(HNS^\varepsilon)\left\{\begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon-\Delta u^\varepsilon &=& -\mathbb{P}\nabla : u^\varepsilon\otimes u^\varepsilon \\ (u^\varepsilon,\partial_t u^\varepsilon)|_{t=0} &=& (u_0^\varepsilon,u_1^\varepsilon)\end{array}\right.$$

which we consider with $(u_0^\varepsilon,u_1^\varepsilon)\in H^{1+\delta}(\mathbb{R}^2)\times H^{\delta}(\mathbb{R}^2)$.
It is more convenient to study the same equation with parameter $\varepsilon=1$:

$$(HNS)\left\{\begin{array}{rcl} \partial_{tt}u+\partial_t u-\Delta u &=& -\mathbb{P}\nabla : u\otimes u \\ (u,\partial_t u)|_{t=0} &=& (u_0,u_1)\in H^{1+\delta}(\mathbb{R}^2)\times H^{\delta}(\mathbb{R}^2)\end{array}\right. .$$

To this purpose, let us set

$$u^\varepsilon(\tau,y)=\frac{1}{\sqrt{\varepsilon}}u\left(\frac{\tau}{\varepsilon},\frac{y}{\sqrt{\varepsilon}}\right). \tag{2.5}$$

This scaling transforms system (HNS^ε) into system (HNS) with initial data

$$\begin{array}{rcl} u_0(x) &=& \sqrt{\varepsilon}u_0^\varepsilon\left(\sqrt{\varepsilon}x\right), \\ u_1(x) &=& \varepsilon^{\frac{3}{2}}\,u_1^\varepsilon\left(\sqrt{\varepsilon}x\right). \end{array}$$

As usual, let us denote by $\Box$ the D'Alembert operator : $\Box=\partial_{tt}-\Delta$ and rewrite (HNS) as follows :

$$(HNS)\left\{\begin{array}{rcl} \Box u &=& -\partial_t u-\mathbb{P}\nabla : u\otimes u =: F(u) \\ (u,\partial_t u)|_{t=0} &=& (u_0,u_1)\in H^{1+\delta}(\mathbb{R}^2)\times H^{\delta}(\mathbb{R}^2)\end{array}\right. .$$

Duhamel's formula for u solution to (HNS) is then

$$u(t)=\cos(t\Lambda)u_0+\frac{\sin(t\Lambda)}{\Lambda}u_1+\int_0^t\frac{\sin\left((t-s)\Lambda\right)}{\Lambda}F\left(u(s)\right)\,\mathrm{d}s=:\phi\left(u\right)(t)\ , \tag{2.6}$$

where Λ is the differentiation operator $\Lambda=\sqrt{-\Delta}$, defined in Fourier variables by

$$\widehat{\Lambda f}(\xi)=|\xi|\hat{f}(\xi).$$

2.2.1.1 Contraction argument

We show local existence for (HNS) in the complete metric space

$$\begin{aligned} X_T(a) &= \Big\{(f,\partial_t f) \in L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)(\mathbb{R}^2)\times L^\infty_T \dot H^\delta(\mathbb{R}^2) : \operatorname{div} f = 0, \\ & \quad \|f\|_{X_T} := \|f\|_{L^\infty_T\dot H^{1+\delta}} + \|f\|_{L^\infty_T \dot H^\delta} + \|\partial_t f\|_{L^\infty_T \dot H^\delta} \le a\Big\}, \end{aligned} \tag{2.7}$$

with $a>0$ and $0<T<1$ to be chosen later. Let $0\le t\le T$.

In the following, we use two facts : first, $\dot H^{1+\delta}\cap \dot H^\delta(\mathbb{R}^2)$ is an algebra (see the book by Alinhac and Gérard [1]) and the inequality

$$\|fg\|_{\dot H^{1+\delta}\cap\dot H^\delta} \le C\left(\|f\|_{\dot H^{1+\delta}}\|g\|_{\dot H^\delta} + \|g\|_{\dot H^{1+\delta}}\|f\|_{\dot H^\delta}\right) \tag{2.8}$$

holds for all $f,g\in \dot H^{1+\delta}\cap \dot H^\delta$. Secondly, since u is divergence-free, one can easily check that

$$\nabla : u\otimes u = u.\nabla u.$$

We want to apply Picard's iteration theorem. So we shall start by looking for suitable reals a and T such that $X_T(a)$ is stable under ϕ.
Let $(u,\partial_t u)\in X_T(a)$. Using Duhamel formula (2.6), we have

$$\begin{aligned} \|\phi(u)(t)\|_{\dot H^{1+\delta}} &\le \|u_0\|_{\dot H^{1+\delta}} + \left\|\frac{\sin(t\Lambda)}{\Lambda}u_1\right\|_{\dot H^{1+\delta}} + \int_0^t \left\|\frac{\sin((t-s)\Lambda)}{\Lambda}F(u(s))\right\|_{\dot H^{1+\delta}} \mathrm{d}s \\ &\le \|u_0\|_{\dot H^{1+\delta}} + \|u_1\|_{\dot H^\delta} + \int_0^t \|F(u(s))\|_{\dot H^\delta}\,\mathrm{d}s \\ &\le \|u_0\|_{\dot H^{1+\delta}} + \|u_1\|_{\dot H^\delta} + \int_0^t \|\partial_t u(s) + \mathbb{P}(u.\nabla u)\|_{\dot H^\delta}\,\mathrm{d}s. \end{aligned}$$

Now let us consider the integral term :

$$\begin{aligned} \int_0^t \|\partial_t u(s) + \mathbb{P}(u.\nabla u)\|_{\dot H^\delta}\,\mathrm{d}s &\le \int_0^t \left(\|\partial_t u(s)\|_{\dot H^\delta} + \|\mathbb{P}(u.\nabla u)\|_{\dot H^\delta}\right)\mathrm{d}s \\ &\le T\left(\|\partial_t u\|_{L^\infty_T\dot H^\delta} + \|\mathbb{P}(u.\nabla u)\|_{L^\infty_T \dot H^\delta}\right). \end{aligned}$$

Using the identity

$$\vec f.\nabla \vec f = \sum_{i=1}^n f_i\partial_i \vec f = \sum_{i=1}^n \partial_i(f_i\vec f), \tag{2.9}$$

that holds for any divergence-free vector $\vec f$ valued in $\mathbb{R}^n$, we have

$$\|\mathbb{P}(u.\nabla u)\|_{\dot H^\delta} \le \sum_{i=1}^2 \|\partial_i(u_i u)\|_{\dot H^\delta} \le \sum_{i=1}^2 \|u_i u\|_{\dot H^{1+\delta}} \le 4C\|u\|^2_{\dot H^{1+\delta}\cap\dot H^\delta}$$

due to inequality (2.8). Thus, we have the following estimate

$$\|\phi(u)(t)\|_{\dot{H}^{1+\delta}} \leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^{\delta}} + Ta\,(1+4Ca)\,.$$

In order to estimate the $\dot{H}^{\delta}(\mathbb{R}^2)$ norm of $\phi(u)(t)$, we use that the operator $\dfrac{\sin(t\Lambda)}{t\Lambda}$ is bounded for all positive t and that its norm is less than 1.

$$\begin{aligned}
\|\phi(u)(t)\|_{\dot{H}^{\delta}} &\leq \|u_0\|_{\dot{H}^{\delta}} + \|\frac{\sin(t\Lambda)}{\Lambda}u_1\|_{\dot{H}^{\delta}} + \int_0^t \left\|\frac{\sin((t-s)\Lambda)}{\Lambda}F(u(s))\right\|_{\dot{H}^{\delta}} \mathrm{d}s \\
&\leq \|u_0\|_{\dot{H}^{\delta}} + T\left\|\frac{\sin(t\Lambda)}{t\Lambda}u_1\right\|_{\dot{H}^{\delta}} + \int_0^t (t-s)\left\|\frac{\sin((t-s)\Lambda)}{(t-s)\Lambda}F(u(s))\right\|_{\dot{H}^{\delta}} \mathrm{d}s \\
&\leq \|u_0\|_{\dot{H}^{\delta}} + T\|u_1\|_{\dot{H}^{\delta}} + \int_0^t (t-s)\|F(u(s))\|_{\dot{H}^{\delta}}\mathrm{d}s.
\end{aligned}$$

Now, let us estimate the integral

$$\begin{aligned}
\int_0^t (t-s)\|F(u(s))\|_{\dot{H}^{\delta}}\,\mathrm{d}s &\leq T\int_0^t \left(\|\partial_t u\|_{\dot{H}^{\delta}} + \|u.\nabla u\|_{\dot{H}^{\delta}}\right)(s)\,\mathrm{d}s \\
&\leq T\int_0^t \left(\|u\|_{X_T} + 4C\|u\|^2_{\dot{H}^{1+\delta}\cap\dot{H}^{\delta}}\right)(s)\,\mathrm{d}s \\
&\leq T^2\left(\|u\|_{X_T} + 4C\|u\|^2_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})}\right).
\end{aligned}$$

Finally,

$$\|\phi(u)(t)\|_{\dot{H}^{\delta}} \leq \|u_0\|_{\dot{H}^{\delta}} + T\|u_1\|_{\dot{H}^{\delta}} + T^2\left(a+4Ca^2\right).$$

Similarly, we have

$$\begin{aligned}
\|\partial_t\left[\phi(u)\right](t)\|_{\dot{H}^{\delta}} &\leq \|-\Lambda\sin(t\Lambda)u_0\|_{\dot{H}^{\delta}} + \|\cos(t\Lambda)u_1\|_{\dot{H}^{\delta}} + \\
&\quad + \int_0^t \|\cos((t-s)\Lambda)\,F(u)(s)\|_{\dot{H}^{\delta}}\,\mathrm{d}s \\
&\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^{\delta}} + T\left(\|\partial_t u\|_{L^\infty_T\dot{H}^{\delta}} + 4C\|u\|^2_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})}\right) \\
&\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^{\delta}} + Ta\,(1+4Ca)\,.
\end{aligned}$$

Finally, we obtain the estimate

$$\|\phi(u)\|_{X_T} \leq 2\left(\|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^{\delta}}\right) + \|u_0\|_{\dot{H}^{\delta}} + T\|u_1\|_{\dot{H}^{\delta}} + aT(2+T)(1+4Ca).$$

Let us set $a = 4\left(\|u_0\|_{\dot{H}^{\delta}} + \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^{\delta}}\right)$ and $T<1$ such that :

$$T\frac{a}{4} + aT(2+T)(1+4Ca) \leq \frac{a}{2}.$$

Since $T<1$, it is sufficient for T to satisfy

$$T \leq \frac{1}{12(1+4Ca)}. \tag{2.10}$$

With a and T chosen that way, the space $X_T(a)$ is stable under ϕ.
It remains to prove that $\phi : X_T(a) \longrightarrow X_T(a)$ is contractive. So let $(u, \partial_t u)$ and $(v, \partial_t v)$ be two elements of $X_T(a)$. Again using Duhamel formula (2.6), we have

$$\begin{aligned}
\left[\phi(u)-\phi(v)\right](t) &= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}\left(\partial_t v + \mathbb{P}(v.\nabla v) - \partial_t u - \mathbb{P}(u.\nabla u)\right)(s)\,\mathrm{d}s \\
&= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}\left(\partial_t (v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\right)(s)\,\mathrm{d}s.
\end{aligned}$$

Let us calculate the X_T norm of $\phi(u)-\phi(v)$.

$$\begin{aligned}
\|\left[\phi(u)-\phi(v)\right](t)\|_{\dot H^{1+\delta}} &\leq \int_0^t \|\partial_t(v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\|_{\dot H^{\delta}}\,\mathrm{d}s \\
&\leq T\left(\|\partial_t(v-u)\|_{L^\infty_T \dot H^\delta} + \|\nabla : (v\otimes v - u\otimes u)\|_{L^\infty_T \dot H^\delta}\right) \\
&\leq T\left(\|v-u\|_{X_T} + \|\nabla : (v\otimes v - u\otimes u)\|_{L^\infty_T \dot H^\delta}\right).
\end{aligned}$$

Now, since u and v are divergence-free, one can check easily that the identity

$$\nabla : (v\otimes v - u\otimes u) = (v-u).\nabla v + u.\nabla(v-u) \tag{2.11}$$

holds and we can write

$$\begin{aligned}
\|\left[\phi(u)-\phi(v)\right](t)\|_{\dot H^{1+\delta}} &\leq T\left(\|v-u\|_{X_T} + \|(v-u).\nabla v\|_{L^\infty_T \dot H^\delta} + \|u.\nabla(v-u)\|_{L^\infty_T \dot H^\delta}\right) \\
&\leq T\Big(\|v-u\|_{X_T} + 4C\|v-u\|_{X_T}\|v\|_{L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)} + \\
&\qquad +4C\|u\|_{L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)}\|v-u\|_{X_T}\Big) \\
&\leq T\left(1 + 4C\|v\|_{L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)} + 4C\|u\|_{L^\infty_T(\dot H^{1+\delta}\cap \dot H^\delta)}\right)\|v-u\|_{X_T} \\
&\leq T\left(1+8Ca\right)\|v-u\|_{X_T}.
\end{aligned}$$

Besides, we estimate the $\dot H^\delta$ norm as follows. Writing

$$\frac{\sin((t-s)\Lambda)}{\Lambda} = (t-s)\frac{\sin((t-s)\Lambda)}{(t-s)\Lambda},$$

we obtain the estimate

$$\begin{aligned}
\|\left[\phi(u)-\phi(v)\right](t)\|_{\dot H^{\delta}} &\leq T\int_0^t \left(\|\partial_t(v-u)\|_{\dot H^\delta} + \|\nabla(v\otimes v - u\otimes u)\|_{\dot H^\delta}\right)(s)\,\mathrm{d}s \\
&\leq T^2\left(\|v-u\|_{X_T} + \|(v-u).\nabla v\|_{L^\infty_T \dot H^\delta} + \|u.\nabla(v-u)\|_{L^\infty_T \dot H^\delta}\right) \\
&\leq T^2\left(1+8Ca\right)\|v-u\|_{X_T}.
\end{aligned}$$

Now, computing $\partial_t\left[\phi(u)-\phi(v)\right](t)$ and estimating its $\dot H^\delta(\mathbb{R}^2)$ norm, we obtain

$$\begin{aligned}\|\partial_t\left[\phi(u)-\phi(v)\right](t)\|_{\dot H^\delta} &\le \int_0^t \|\cos\left((t-s)\Lambda\right)\left(\partial_t(v-u)+\mathbb{P}\nabla:(v\otimes v-u\otimes u)\right)\|_{\dot H^\delta}(s)\,\mathrm{d}s\\ &\le T\left(\|\partial_t(v-u)\|_{L^\infty_T\dot H^\delta}+\|(v-u).\nabla v\|_{L^\infty_T\dot H^\delta}+\|u.\nabla(v-u)\|_{L^\infty_T\dot H^\delta}\right)\\ &\le T\left(1+8Ca\right)\|v-u\|_{X_T}.\end{aligned}$$

By similar computations and using (2.11), we have the estimate

$$\|\phi(u)-\phi(v)\|_{X_T}\le T(2+T)\left(1+8Ca\right)\|v-u\|_{X_T}$$

so ϕ is contractive if

$$T(2+T)<\frac{1}{(1+8Ca)},$$

or, since $T<1$, if

$$T<\frac{1}{3(1+8Ca)}. \tag{2.12}$$

Combining conditions (2.10) and (2.12) above, we deduce the local existence of a solution to (HNS), defined on $[0,T[$, for all positive T such that

$$T\le\frac{1}{12+48C\left(\|u_0\|_{\dot H^{1+\delta}(\mathbb{R}^2)}+\|u_0\|_{\dot H^\delta(\mathbb{R}^2)}+\|u_1\|_{\dot H^\delta(\mathbb{R}^2)}\right)}. \tag{2.13}$$

In particular, as long as $\|u(t)\|_X:=\|u(t)\|_{\dot H^{1+\delta}(\mathbb{R}^2)}+\|u(t)\|_{\dot H^\delta(\mathbb{R}^2)}+\|\partial_t u(t)\|_{\dot H^\delta(\mathbb{R}^2)}$ remains bounded, we can iterate the fixed point argument and extend the solution.

2.2.1.2 Globalization

Let us resume the initial equation (HNS^ε). One can easily check that

$$\|u(t)\|_X=\varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot H^{1+\delta}(\mathbb{R}^2)}+\|u^\varepsilon\|_{\dot H^\delta(\mathbb{R}^2)}+\varepsilon\|\partial_t u^\varepsilon\|_{\dot H^\delta(\mathbb{R}^2)}\right).$$

Then, for all non negative real δ and all non negative t, we define the energy

$$E^\delta_\varepsilon(t)=\int_{\mathbb{R}^2}\left(\frac{1}{2}|\Lambda^\delta(u^\varepsilon+\varepsilon\partial_t u^\varepsilon)|^2+\frac{\varepsilon^2}{2}|\Lambda^\delta\partial_t u^\varepsilon|^2+\varepsilon|\Lambda^{1+\delta}u^\varepsilon|^2\right)\mathrm{d}x$$

so that we have

$$\varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot H^{1+\delta}(\mathbb{R}^2)}+\|u^\varepsilon\|_{\dot H^\delta(\mathbb{R}^2)}+\varepsilon\|\partial_t u^\varepsilon\|_{\dot H^\delta(\mathbb{R}^2)}\right)\le C\varepsilon^{\frac{\delta}{2}}\sqrt{E^\delta_\varepsilon}.$$

In this subsection, we shall prove the following inequality that yields the globality of the solution, *i.e.* $T_\varepsilon^{\max} = +\infty$, where $T_\varepsilon^{\max}$ is the maximal existence time for the initial value problem (HNS^ε) :

$$E_\varepsilon^\delta(t) \leq C_0 \varepsilon^{-\delta} \tag{2.14}$$

for all $t > 0$, where C_0 is a universal constant.

First, let us point out that $\dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$ is an algebra and that the product estimate

$$\|fg\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} \leq C_1 \left(\|f\|_{\dot{H}^{1+\delta}} \|g\|_\infty + \|g\|_{\dot{H}^{1+\delta}} \|f\|_\infty \right). \tag{2.15}$$

holds (see [1]) for all functions $f, g \in \dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$. Moreover, we know that the homogeneous Besov [1] space $\dot{B}^1_{2,1}(\mathbb{R}^2)$ embeds into $L^\infty(\mathbb{R}^2)$ and, interpolating, we obtain

$$\|f\|_\infty \ \leq \ \tilde{C} \|f\|_{\dot{B}^1_{2,1}} \ \leq \ C_2 \, \|f\|^{\delta}_{\dot{H}^\delta} \, . \, \|f\|^{1-\delta}_{\dot{H}^{1+\delta}} \, . \tag{2.16}$$

Lemma 2.1. *Assume the following, when ε goes to zero :*

$$(H) \left\{ \begin{array}{ll} i) & \varepsilon^{\frac{1+\delta}{2}} \|u_0^\varepsilon\|_{\dot{H}^{1+\delta}} + \varepsilon^{\frac{\delta}{2}} \|u_0^\varepsilon\|_{\dot{H}^\delta} = o(1) \\ ii) & \varepsilon^{\frac{1}{2}} \|u_0^\varepsilon\|_{\dot{H}^1} + \varepsilon \|u_1^\varepsilon\|_{L^2} = o(1). \end{array} \right.$$

Let us define $0 \leq T \leq T_\varepsilon^{\max}$ by

$$T = \sup \left\{ 0 \leq \tau \leq T_\varepsilon^{\max} \ : \ \forall \ t \in [0, \tau[, \ \|u^\varepsilon(t)\|_\infty < \frac{1}{8 \ C_1 \sqrt{\varepsilon}} \right\}.$$

Then, for ε small enough, there exists a large number N, only depending on δ and $\|u_0^\varepsilon\|_{L^2}$ (which is arbitrary), such that, for all $0 \leq t \leq T$,

$$E_\varepsilon^\delta(t) \leq E_\varepsilon^\delta(0) \left(2 \|u_0^\varepsilon\|^2_{L^2} + 1 \right)^N. \tag{2.17}$$

Proof. Let us compute the time derivative of E_ε^δ.

$$\begin{aligned} \frac{dE_\varepsilon^\delta}{dt} &= \int \Big(\Lambda^\delta(\varepsilon \partial_{tt} u^\varepsilon + \partial_t u^\varepsilon) . \Lambda^\delta(u^\varepsilon + \varepsilon \partial_t u^\varepsilon) + \varepsilon^2 \Lambda^\delta(\partial_t u^\varepsilon) . \Lambda^\delta(\partial_{tt} u^\varepsilon) + \\ &\qquad + 2\varepsilon \Lambda^{\delta+1}(\partial_t u^\varepsilon) . \Lambda^{\delta+1} u^\varepsilon \Big) \, \mathrm{d}x \\ &= \int \Big(\Lambda^\delta(\varepsilon \partial_{tt} u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon) . \Lambda^\delta(u^\varepsilon + 2\varepsilon \partial_t u^\varepsilon) - \varepsilon |\Lambda^\delta \partial_t u^\varepsilon|^2 - |\Lambda^{\delta+1} u^\varepsilon|^2 \Big) \, \mathrm{d}x \\ &= \int \Big(- \Lambda^\delta(u^\varepsilon . \nabla u^\varepsilon) . \Lambda^\delta u^\varepsilon - 2\varepsilon \Lambda^\delta(u^\varepsilon . \nabla u^\varepsilon) . \Lambda^\delta \partial_t u^\varepsilon - \varepsilon |\Lambda^\delta \partial_t u^\varepsilon|^2 - |\Lambda^{\delta+1} u^\varepsilon|^2 \Big) \, \mathrm{d}x \end{aligned}$$

1. For definitions and properties of the Besov spaces, see the book by P.-G. Lemarié-Rieusset [28]

Performing a classical Young inequality $2ab \leq a^2 + b^2$ and rearranging the terms, we obtain

$$\frac{dE_\varepsilon^\delta}{dt}(t) \leq -\int \Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon \,\mathrm{d}x + \int \left(\varepsilon|\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon)|^2 - |\Lambda^{\delta+1}u^\varepsilon|^2\right) \mathrm{d}x$$

Now, let us recall that

$$u^\varepsilon.\nabla u^\varepsilon = \sum_{i=1}^{2} u_i^\varepsilon \partial_i u^\varepsilon = \sum_{i=1}^{2} \partial_i(u_i^\varepsilon u^\varepsilon)$$

since u^ε is divergence-free. Now, (2.15) yields

$$\|u^\varepsilon.\nabla u^\varepsilon\|_{\dot{H}^\delta} \leq \sum_{i=1}^{2} \|\partial_i(u_i^\varepsilon u^\varepsilon)\|_{\dot{H}^\delta} \leq 4C_1\|u^\varepsilon\|_\infty \|u^\varepsilon\|_{\dot{H}^{1+\delta}}$$

Thus

$$\begin{aligned}\left|\int \Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon \,\mathrm{d}x\right| &\leq \|u^\varepsilon.\nabla u^\varepsilon\|_{\dot{H}^\delta}\|u^\varepsilon\|_{\dot{H}^\delta} \\ &\leq 4C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot{H}^{1+\delta}}\|u^\varepsilon\|_{\dot{H}^\delta} \\ &\leq 4C_1C_2\|u^\varepsilon\|_{\dot{H}^\delta}^\delta\|u^\varepsilon\|_{\dot{H}^\delta}\|u^\varepsilon\|_{\dot{H}^{1+\delta}}^{2-\delta}.\end{aligned}$$

The interpolation inequality

$$\|u^\varepsilon\|_{\dot{H}^\delta} \leq C_3\|u^\varepsilon\|_2^{1-\delta}\|u^\varepsilon\|_{\dot{H}^1}^\delta$$

yields finally

$$\left|\int \Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon \,dx\right| \leq 4C_1C_2C_3\|u^\varepsilon\|_2^{1-\delta}\left(\|u^\varepsilon\|_{\dot{H}^1}\|u^\varepsilon\|_{\dot{H}^\delta}\right)^\delta\|u^\varepsilon\|_{\dot{H}^{1+\delta}}^{2-\delta}.$$

Besides, since we assume $i)$ and using inequality (2.16), we can write, for ε small enough,

$$\|u_0^\varepsilon\|_\infty \leq \frac{1}{8C_1\sqrt{\varepsilon}}. \tag{2.18}$$

Now, by continuity of the (local) solution u^ε with respect to t, we deduce $T > 0$ and the inequality

$$\begin{aligned}\int \left(\varepsilon|\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon)|^2 - |\Lambda^{\delta+1}u^\varepsilon|^2\right) \mathrm{d}x &\leq \left(16C_1^2\varepsilon\|u^\varepsilon\|_\infty^2 - 1\right)\int |\Lambda^{\delta+1}u^\varepsilon|^2 \,\mathrm{d}x \\ &\leq -\frac{1}{4}\|u^\varepsilon\|_{\dot{H}^{1+\delta}}^2 \end{aligned} \tag{2.19}$$

holds on $[0, T)$.
Hence, for all $0 \le t < T$,

$$\frac{d}{dt}E_\varepsilon^\delta(t) \;\le\; -\frac{1}{4}\|u^\varepsilon\|_{\dot{H}^{1+\delta}}^2 + 4C_1C_2C_3\|u^\varepsilon\|_2^{1-\delta}\left(\|u^\varepsilon\|_{\dot{H}^1}\|u^\varepsilon\|_{\dot{H}^\delta}\right)^\delta \|u^\varepsilon\|_{\dot{H}^{1+\delta}}^{2-\delta}.$$

Now, consider the second term on the right hand side and use Young inequality

$$ab \le \frac{\delta}{2}\, a^{\frac{2}{\delta}} + \frac{2-\delta}{2}\, b^{\frac{2}{2-\delta}}$$

with $b = 2^{\delta-2}\|u^\varepsilon\|_{\dot{H}^{1+\delta}}^{2-\delta}$. We obtain

$$\frac{d}{dt}E_\varepsilon^\delta(t) \;\le\; \frac{\delta}{2}\, 2^{2\frac{2-\delta}{\delta}}(4C_1C_2C_3)^{\frac{2}{\delta}}\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\|u^\varepsilon\|_{\dot{H}^1}^2\|u^\varepsilon\|_{\dot{H}^\delta}^2.$$

Finally, the inequality

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2 \ge 2\|b\|^2$$

with $a = \dfrac{u^\varepsilon}{2} + \varepsilon\partial_t u^\varepsilon$ and $b = \dfrac{u^\varepsilon}{2}$ yields, for all non negative δ, the estimate

$$E_\varepsilon^\delta(t) \;\ge\; \|u^\varepsilon + \varepsilon\partial_t u^\varepsilon\|_{\dot{H}^\delta}^2 + \|\varepsilon\partial_t u^\varepsilon\|_{\dot{H}^\delta}^2 \;\ge\; \frac{1}{2}\|u^\varepsilon\|_{\dot{H}^\delta}^2 \qquad (2.20)$$

from which we deduce

$$\frac{d}{dt}E_\varepsilon^\delta(t) \le \delta\; 2^{2\frac{2-\delta}{\delta}}(4C_1C_2C_3)^{\frac{2}{\delta}}\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\|u^\varepsilon\|_{\dot{H}^1}^2 E_\varepsilon^\delta(t).$$

Instead of showing that $t \mapsto E_\varepsilon^\delta(t)$ is decreasing, we will prove that $E^\delta(E_\varepsilon^0+1)^N$ decreases if N is large enough. Even though this new energy does not seem natural, it is more convenient than $E^\delta + NE_\varepsilon^0$ since the latter would have required a smallness assumption on the L^2 norm of u_0^ε. But this would yield a weaker result than one might hope as the Navier-Stokes equations with initial data in $L^2(\mathbb{R}^2)$ unconditionally has a global solution.

First of all, let us notice that $\int_{\mathbb{R}^n} f.(f.\nabla f) = 0$ for all divergence-free function f and that (see inequality (2.19))

$$\frac{dE_\varepsilon^0}{dt}(t) \;\le\; -\frac{1}{4}\|u^\varepsilon(t)\|_{\dot{H}^1}^2.$$

We use it in (2.21) below.

$$\begin{aligned}\frac{d}{dt}\left[(E_\varepsilon^0+1)^N E_\varepsilon^\delta\right] &= N\frac{dE_\varepsilon^0}{dt}(E_\varepsilon^0+1)^{N-1}E_\varepsilon^\delta + (E_\varepsilon^0+1)^N\frac{dE_\varepsilon^\delta}{dt}\\ &\leq \frac{1}{4}\,(E_\varepsilon^0+1)^{N-1}E_\varepsilon^\delta\|u^\varepsilon\|_{\dot H^1}^2\times\\ &\quad\times\left[-N+\delta\,(16C_1C_2C_3)^{\frac{2}{\delta}}(E_\varepsilon^0+1)\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\right]. \qquad (2.21)\end{aligned}$$

Under the assumption $ii)$, we have

$$\begin{aligned}E_\varepsilon^0(0) &= \int_{\mathbb{R}^2}\left(\frac{|u_0^\varepsilon+\varepsilon u_1^\varepsilon|^2+\varepsilon^2|u_1^\varepsilon|^2}{2}+\varepsilon|\nabla u_0^\varepsilon|^2\right)\mathrm{d}x\\ &\leq \|u_0^\varepsilon\|_2^2+\frac{3\varepsilon^2}{2}\|u_1^\varepsilon\|_2^2+\varepsilon\|u_0^\varepsilon\|_{\dot H^1}^2\\ &< 2\,\|u_0^\varepsilon\|_2^2\end{aligned}$$

if ε is small enough. Now, (2.20) with $\delta=0$ yields

$$\|u^\varepsilon(t)\|_2^2 \;\leq\; 2E_\varepsilon^0(t) \;<\; 4\|u_0^\varepsilon\|_2^2 \qquad (2.22)$$

for all $t\in(0,T)$. We have obtained

$$\frac{d}{dt}\left[(E_\varepsilon^0+1)^N E_\varepsilon^\delta\right] \leq \frac{1}{4}(E_\varepsilon^0+1)^{N-1}E_\varepsilon^\delta\|u^\varepsilon\|_{\dot H^1}^2\left[-N+K_\delta(4\|u_0^\varepsilon\|_2^2)^{\frac{1-\delta}{\delta}}(2\|u_0^\varepsilon\|_2^2+1)\right].$$

We deduce that, for N large enough (depending only on δ and $\|u_0^\varepsilon\|_2$), the right hand side is negative and

$$E_\varepsilon^\delta(t)\leq E_\varepsilon^\delta(0)(E_\varepsilon^0(0)+1)^N < E_\varepsilon^\delta(0)(2\|u_0^\varepsilon\|_2^2+1)^N$$

for all $0\leq t<T$. □

We want to reiterate the reasoning, *i.e.* to keep the control of $\|u^\varepsilon(t)\|_\infty$. The aim of Lemma 2.2 is to ensure this control throughout the time.

Lemma 2.2. *Assume the limit*

$$\varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot H^\delta}\longrightarrow 0\ ,\quad \varepsilon\to 0 \qquad (2.23)$$

in addition to the assumptions (H) in Lemma 2.1. Then the inequality

$$E_\varepsilon^\delta(0) \;<\; \frac{2^{-\delta}}{(16C_1C_2)^2}\,\varepsilon^{-\delta}(2\|u_0^\varepsilon\|_2^2+1)^{-N}. \qquad (2.24)$$

holds for ε small enough. Moreover, for all $t \in [0,T)$,

$$\|u^\varepsilon\|_\infty \leq \frac{1}{16C_1\sqrt{\varepsilon}}. \tag{2.25}$$

Proof. Easy calculations show that (2.24) is true as a result of (H) and (2.23). Now, let $t \in [0,T)$ and recall interpolation inequality (2.16)

$$\|u^\varepsilon\|_\infty \leq C_2\|u^\varepsilon\|_{\dot{H}^\delta}^\delta \|u^\varepsilon\|_{\dot{H}^{1+\delta}}^{1-\delta}.$$

Due to inequality (2.20), we have

$$\|u^\varepsilon(t)\|_\infty \leq C_2\sqrt{2E_\varepsilon^\delta(t)}^\delta \sqrt{\varepsilon^{-1}E_\varepsilon^\delta(t)}^{1-\delta} = C_2 2^{\frac{\delta}{2}}\varepsilon^{\frac{\delta-1}{2}}\sqrt{E_\varepsilon^\delta(t)}.$$

Using (2.17) (conclusion of Lemma 2.1) and (2.24), we obtain

$$\|u^\varepsilon(t)\|_\infty \leq C_2 2^{\frac{\delta}{2}}\varepsilon^{\frac{\delta-1}{2}}\frac{2^{-\frac{\delta}{2}}}{16C_1C_2}\varepsilon^{-\frac{\delta}{2}} = \frac{1}{16C_1}\varepsilon^{-\frac{1}{2}}$$

for all $0 < t < T$. □

Remark. *Under the assumptions* (2.1) *in Theorem 2.1, the conditions* (H) *in Lemma 2.1 and* (2.23) *in Lemma 2.2 are fulfilled.*

Now, we shall prove that these estimates (on $\|u^\varepsilon(t)\|_\infty$ and, consequently, on $E_\varepsilon^\delta(t)$) remain true on the whole existence interval $[0, T_\varepsilon^{\max})$, where $T_\varepsilon^{\max}$ is the existence time (2.13) given by the Picard iteration. We have already proved that $T > 0$. Assume $T < T_\varepsilon^{\max}$. Then

$$\|u^\varepsilon(T)\|_\infty = \frac{1}{8C_1\sqrt{\varepsilon}}. \tag{2.26}$$

On the other hand, (2.25) in Lemma 2.2 yields

$$\|u^\varepsilon(T)\|_\infty \leq \frac{1}{16C_1\sqrt{\varepsilon}} < \frac{1}{8C_1\sqrt{\varepsilon}},$$

which contradicts (2.26) so $T \geq T_\varepsilon^{\max}$. We deduce then from Lemma 2.1 and Lemma 2.2 that E_ε^δ satisfies inequality (2.14) on the existence interval $[0, T_\varepsilon^{\max})$. Therefore the (HNS^ε) equation has a global solution.

2.2.2 Convergence towards a solution to the Navier-Stokes problem

Let us recall that Brenier, Natalini and Puel showed in [7] that, under suitable assumptions, the solutions to (HNS^ε), with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^2 \times H^1(\mathbb{T}^2)$, converge to the solutions to (NS) with smooth initial data v_0 when ε goes to 0. The authors used the modulated energy method (see [7] and references therein) to show an error estimate in the $L^\infty_T L^2(\mathbb{T}^2)$ norm, for all positive T.

In this section, we shall prove a similar result with less regularity on the initial data $(u_0^\varepsilon, u_1^\varepsilon)$ and v_0, in $\mathbb{R}^2$ instead of $\mathbb{T}^2$. Indeed, we prove that, under suitable assumptions (less restrictive than those in [7]) and for any positive δ, the solutions to (HNS^ε), with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta} \times H^\delta(\mathbb{R}^2)$, converge in the $L^2(\mathbb{R}^2)^2$ norm, when ε goes to 0, to the solutions to (NS) with initial data $v_0 \in \dot{H}^s(\mathbb{R}^2)^2$, $s > 0$. We recall here that the critical space for the Navier-Stokes equations in $\mathbb{R}^2$ is L^2.
Because of the loss of regularity, the proof is not straightforward anymore. Interpolation and product estimates will help to get around the difficulties.

We shall start the proof like in [7], introducing the so-called Dafermos modulated energy, which is the total energy of the wave equation (HNS^ε), modulated by a divergence-free vector $(t, x) \mapsto v(t, x)$

$$E_{\varepsilon,v}(t) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|u^\varepsilon - v(t,x) + \varepsilon \partial_t u^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon|^2 + \varepsilon |\nabla u^\varepsilon|^2 \right) \mathrm{d}x. \tag{2.27}$$

This energy satisfies the inequality

$$\int_{\mathbb{R}^2} |u^\varepsilon - v|^2 \,\mathrm{d}x \leq 4 E_{\varepsilon,v}(t). \tag{2.28}$$

Via a Gronwall inequality, we shall show that, for all positive t such that $t < T$, the modulated energy $E_{\varepsilon,v}(t)$ converges to 0 when ε goes to 0. To this end, let us compute the time derivative of $E_{\varepsilon,v}$.

Lemma 2.3 (see [7]). *If u^ε and v are divergence-free functions, the Dafermos modulated energy satisfies the identity*

$$\begin{aligned} \frac{d}{dt} E_{\varepsilon,v}(t) \quad = \quad & \int v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v)\, dx - \varepsilon \int |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \, dx \\ & -\varepsilon \int \partial_t v . \partial_t u^\varepsilon \, dx + \int \left(\varepsilon |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 - |\nabla (u^\varepsilon - v)|^2 \right) dx \\ & + \int (\partial_t v + v.\nabla v - \Delta v)(v - u^\varepsilon)\, dx. \end{aligned} \tag{2.29}$$

Remark. *This lemma is proved in [7].*

From now on, let v be a solution to the initial value problem

$$\partial_t v = \Delta v - \mathbb{P}\nabla : v \otimes v \ , \quad \mathrm{div}\, v = 0 \ , \quad v|_{t=0} = v_0 \in H^s(\mathbb{R}^2), \tag{2.30}$$

where s is a positive real, so that the last term of (2.29) vanishes. Now, since the second term in the identity (2.29) is negative, (2.29) writes

$$\begin{aligned} \frac{d}{dt} E_{\varepsilon,v}(t) \quad &\leq \quad -\varepsilon \int_{\mathbb{R}^2} \partial_t v . \partial_t u^\varepsilon \, \mathrm{d}x + \int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v) \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} \left(\varepsilon |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 - |\nabla (u^\varepsilon - v)|^2 \right) \, \mathrm{d}x. \end{aligned} \tag{2.31}$$

We shall treat the three terms on the right hand side in the following three subsections.

First, let us recall Duhamel's formula for the Navier-Stokes equations (2.30) :

$$v(t,.) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla : (v \otimes v)(s,.) \, \mathrm{d}s. \tag{2.32}$$

Computing the time derivative of (2.32), we obtain

$$\partial_t v(t,.) = \Delta e^{t\Delta} v_0 - \mathbb{P}\nabla : (v \otimes v)(t,.) - \int_0^t \Delta e^{(t-s)\Delta} \mathbb{P}\nabla : (v \otimes v)(s,.) \, \mathrm{d}s. \tag{2.33}$$

Remark. *If $v_0 \in H^s(\mathbb{R}^2)$, then, for all positive T, the solution v to* (2.30) *satisfies*

$$v \in L^2_T H^{s+1}(\mathbb{R}^2) \cap L^\infty_T H^s(\mathbb{R}^2).$$

2.2.2.1 Estimating $\varepsilon \int_0^T \int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v \, \mathbf{d}t \, \mathbf{d}x$

In order to bound the integral by a positive power of ε, we shall find spaces $L^p_T \dot{H}^\sigma(\mathbb{R}^2)$ and $L^q_T \dot{H}^{\sigma'}(\mathbb{R}^2)$ containing respectively $\partial_t v$ and $\varepsilon \partial_t u^\varepsilon$.

Step 1 : Regularity of $\partial_t v$.

Lemma 2.4. *Let $\theta \in [0,1[$ and v be a solution to equation* (2.30)*. Then, for all $0 < \tilde{\varepsilon} < s$,*

$$\partial_t v \in L^p_T \dot{H}^\sigma(\mathbb{R}^2), \tag{2.34}$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{2} + \theta = \dfrac{1+\theta}{2}$ *and* $\sigma = (1-\theta)(s-1) + \theta(s-\tilde{\varepsilon}) = s - 1 + \theta(1-\tilde{\varepsilon})$.

Proof. Recall that

$$\partial_t v(t,.) = \Delta e^{t\Delta} v_0 - \nabla : (v \otimes v)(t,.) - \int_0^t \Delta e^{(t-s)\Delta} \nabla : (v \otimes v)(s,.)\,\mathrm{d}s.$$

First, standard estimates yield

$$\Delta e^{t\Delta} v_0 \in L^2_T \dot{H}^{s+1} \cap L^1_T \dot{H}^{s-\tilde{\varepsilon}}.$$

Let us consider now the term $\nabla : (v \otimes v)$. Using remark 2.2.2 and a classical Sobolev embedding, we have $v \in L^2_T L^\infty \cap L^\infty_T H^s$. Since $L^\infty \cap \dot{H}^s$ is an algebra (see [1] page 98), we deduce that

$$(v \otimes v)_{i,j} \;=\; v_i v_j \in L^2_T \dot{H}^s$$

Consequently,

$$\nabla : (v \otimes v) \in L^2_T \dot{H}^{s-1}.$$

Moreover, using that $v \in L^2_T H^{s+1}(\mathbb{R}^2)$ and $H^{s+1}(\mathbb{R}^2)$ being an algebra (see [1]), we obtain that $(v \otimes v)_{i,j} \;=\; v_i v_j \in L^1_T H^{s+1}(\mathbb{R}^2)$, so that

$$\nabla : (v \otimes v) \in L^1_T H^s(\mathbb{R}^2).$$

Interpolating between $L^2_T \dot{H}^{s-1}(\mathbb{R}^2)$ and $L^1_T \dot{H}^{s-\tilde{\varepsilon}}$ and taking $\theta \in [0,1]$, we have

$$\Delta e^{t\Delta} v_0 \;, \quad \mathbb{P}\nabla : (v \otimes v) \in L^{\frac{2}{1+\theta}} \dot{H}^{s-1+\theta(1-\tilde{\varepsilon})}.$$

The following theorem (proved in [28] page 64), applied to $\mathbb{P}\nabla : (v \otimes v)$, allows us to conclude immediately that the integral term is also in $L^{\frac{2}{1+\theta}} \dot{H}^{s-1+\theta(1-\tilde{\varepsilon})}$ for all $\theta \in [0,1)$.

Theorem 2.3 (Maximal $L^p L^q$ regularity for the heat kernel)**.** *Let A be an operator defined by*

$$f(t,x) \mapsto Af(t,x) = \int_0^t e^{(t-s)\Delta} \Delta f(s,x)\, ds.$$

Then A is bounded from $L^p_T L^q(\mathbb{R}^d)$ to itself, for all reals $T > 0$, $1 < p < \infty$ and $1 < q < \infty$.

□

Step 2 : Regularity of $\varepsilon\partial_t u^\varepsilon$.

Lemma 2.5. *Let $0 \le \theta \le 1$ and u^ε be a solution to (HNS^ε). Then*

$$\varepsilon\partial_t u^\varepsilon \in L^q_T \dot{H}^{\sigma'}(\mathbb{R}^2) \tag{2.35}$$

where $\dfrac{1}{q} = \dfrac{1-\theta}{2}$ *and* $\sigma' = \theta\delta$ *and*

$$\|\varepsilon\partial_t u^\varepsilon\|_{L^q_T \dot{H}^{\sigma'}} = \mathcal{O}\left(\varepsilon^{\frac{1-\theta}{2}-\frac{\theta\delta}{2}}\right).$$

Proof. First, let us show that

$$\varepsilon\partial_t u^\varepsilon \in L^2_T L^2(\mathbb{R}^2, \varepsilon^{-1/2}\mathrm{d}x) \cap L^\infty_T \dot{H}^\delta(\mathbb{R}^2, \varepsilon^{\delta/2}\mathrm{d}x). \tag{2.36}$$

To this purpose, let us consider the energy

$$E^0_\varepsilon(t) =: \int_{\mathbb{R}^2} \left(\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon|^2 + \varepsilon|\nabla u^\varepsilon|^2\right) \mathrm{d}x$$

and differentiate it. A simple calculation gives us

$$\frac{d}{dt}E^0_\varepsilon(t) + \varepsilon\int_{\mathbb{R}^2} |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\,\mathrm{d}x + \int_{\mathbb{R}^2}\left(|\nabla u^\varepsilon|^2 - \varepsilon|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2\right)\mathrm{d}x = 0.$$

Due to the control of the norm $\|u^\varepsilon\|_\infty$ by $\dfrac{1}{\sqrt{2\varepsilon}}$, the last term in the left hand side is bounded from below by $\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,dx$ and we obtain

$$E^0_\varepsilon(T) + \int_0^T\int_{\mathbb{R}^2} \varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon\otimes u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le E^0_\varepsilon(0) \le C_0.$$

Therefore, we have that $\int_0^T\int_{\mathbb{R}^2}\varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon\otimes u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t \le C$ and

$$\int_0^T\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le C$$

which gives

$$\varepsilon\int_0^T\int_{\mathbb{R}^2}|\nabla : (u^\varepsilon\otimes u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon\|u^\varepsilon\|_\infty\int_0^T\int_{\mathbb{R}^2}|\nabla u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Consequently,

$$\begin{aligned}\|\sqrt{\varepsilon}\partial_t u^\varepsilon\|^2_{L^2_T L^2} &\le 2\varepsilon\int_0^T\int_{\mathbb{R}^2}\left(|\partial_t u^\varepsilon + \nabla{:}(u^\varepsilon\otimes u^\varepsilon)|^2 + |\nabla{:}(u^\varepsilon\otimes u^\varepsilon)|^2\right)\mathrm{d}x\,\mathrm{d}t\\ &\le 2C + 2C = 4C,\end{aligned}$$

i.e. $\sqrt{\varepsilon}\partial_t u^\varepsilon \in L^2_T L^2$ uniformly in parameter ε.

Moreover, directly from the expression of E^δ_ε and from (2.14), we deduce that

$$\|\partial_t u^\varepsilon\|_{L^\infty_T \dot{H}^\delta(\mathbb{R}^2)} = \mathcal{O}\left(\varepsilon^{-1-\frac{\delta}{2}}\right).$$

Now, interpolating between the two spaces in (2.36), we obtain (2.35). □

We will now choose un appropriate $0 < \theta < 1$, such that

$$\varepsilon \int_0^T \int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v \,\mathrm{d}t\,\mathrm{d}x \longrightarrow 0\ , \quad \varepsilon \to 0.$$

So we should have

$$-1 + \theta(1+\delta) < 0 \Longleftrightarrow 0 < \theta < \frac{1}{1+\delta}, \tag{2.37}$$

so that the power of ε controlling the integral is negative. Moreover, $L^q_T H^{\sigma'}$ is the dual space of $L^p_T H^\sigma$ if $\sigma' = -\sigma$, *i.e.*

$$\theta = \frac{1-s}{1+\delta-\tilde{\varepsilon}}.$$

This value of θ satisfies condition (2.37) and gives us

$$\varepsilon \int_0^T \int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v \,\mathrm{d}t\,\mathrm{d}x = \mathcal{O}\left(\sqrt{\varepsilon}^{(s(1+\delta)-\tilde{\varepsilon})/(1+\delta-\tilde{\varepsilon})}\right) = \mathcal{O}\left(\varepsilon^\nu\right),$$

with $\nu = \dfrac{s(1+\delta)-\tilde{\varepsilon}}{2(1+\delta-\tilde{\varepsilon})} > 0$. Notice that $0 < \nu < \frac{s}{2}$ and that ν can be chosen arbitrarily close to $\frac{s}{2}$.

2.2.2.2 Estimating $\int_0^T\int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v)\,\mathrm{d}t\,\mathrm{d}x$

First, let us set

$$I = \int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v)\,\mathrm{d}t\,\mathrm{d}x.$$

Then we prove the following lemma :

Lemma 2.6. *Let v and u^ε be as above. We have*

$$\int_0^T I(t)\,dt \leq 2\int_0^T \|v\|_{BMO}^2 E_{\varepsilon,v}(t)\,dt + \frac{1}{2}\int_0^T \|\nabla(u^\varepsilon - v)\|_{L^2}^2\,dt. \tag{2.38}$$

Proof. Let us recall that $v(t) \in \dot{H}^1(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$ (see remark 2.2.2) and that the solution $u^\varepsilon(t)$ of the wave equation (HNS^ε) is an $L^2(\mathbb{R}^2)$ divergence-free vector. The following theorem applies.

Theorem 2.4 (div-curl lemma, see [13])**.** *Let f be an $L^2(\mathbb{R}^2)$ divergence-free vector and $g \in \dot{H}^1(\mathbb{R}^2)$. Then*

$$f.\nabla g \in \mathcal{H}^1(\mathbb{R}^2),$$

where $\mathcal{H}^1(\mathbb{R}^2)$ is the Hardy space constructed on $L^1(\mathbb{R}^2)$.

Using the duality of Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and $BMO(\mathbb{R}^2)$ (proved in [15]), we write

$$I \leq c'\|v\|_{BMO}\|\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\|_{\mathcal{H}^1},$$

where c' is a universal constant. Then the div-curl theorem (Theorem 2.4) applied to the $\mathcal{H}^1$ norm in the right hand side yields

$$\begin{aligned} I &\leq c\|v\|_{BMO}\|u^\varepsilon - v\|_{L^2}\|\nabla(u^\varepsilon - v)\|_{L^2} \\ &\leq \frac{c^2}{2}\|v\|_{BMO}^2\|u^\varepsilon - v\|_{L^2}^2 + \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \\ &\overset{(2.28)}{\leq} 2c^2\|v\|_{BMO}^2 E_{\varepsilon,v}(t) + \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \end{aligned}$$

by a Young inequality. Therefore, we have proved (2.38). □

At this level, we have the following inequality

$$\begin{aligned} E_{\varepsilon,v}(T) \leq\ & E_{\varepsilon,v}(0) + 2c^2\int_0^T \|v\|_{BMO}^2 E_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}(\varepsilon^\nu) + \\ & + \int_0^T \left(\varepsilon\|\nabla : (u^\varepsilon\otimes u^\varepsilon)\|_{L^2}^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2\right)\mathrm{d}t. \end{aligned}$$

Let us call $\tilde{A}^\varepsilon$ the term that remains to estimate :

$$\tilde{A}^\varepsilon = \int_0^T A^\varepsilon\,\mathrm{d}t = \int_0^T \left(\varepsilon\|\nabla : (u^\varepsilon\otimes u^\varepsilon)\|_{L^2}^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2\right)\mathrm{d}t.$$

2.2.2.3 Estimating $\int_0^T A^\varepsilon \, \mathbf{d}t$

The aim of this part is to prove the following lemma

Lemma 2.7. *Let v and u^ε be as above. Then, for all positive T and for some positive μ, we have*

$$\int_0^T A^\varepsilon \, dt = \int_0^T \int_{\mathbb{R}^2} \left(\varepsilon |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 - \frac{1}{2} |\nabla(u^\varepsilon - v)|^2 \right) \, dt \, dx = \mathcal{O}(\varepsilon^\mu).$$

Proof. First, let us write

$$\nabla : (u^\varepsilon \otimes u^\varepsilon) = u^\varepsilon . \nabla u^\varepsilon = u^\varepsilon . \nabla (u^\varepsilon - v) + u^\varepsilon . \nabla v$$

then perform a Young inequality

$$|\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \leq 2 |u^\varepsilon . \nabla (u^\varepsilon - v)|^2 + 2 |u^\varepsilon . \nabla v|^2.$$

So we have that

$$\begin{aligned}
A^\varepsilon &= \varepsilon \int_{\mathbb{R}^2} |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \, \mathrm{d}x - \frac{1}{2} \|\nabla(u^\varepsilon - v)\|^2_{L^2} \\
&\leq 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon . \nabla(u^\varepsilon - v)|^2 \, \mathrm{d}x + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon . \nabla v|^2 \, \mathrm{d}x - \frac{1}{2} \|\nabla(u^\varepsilon - v)\|^2_{L^2} \\
&\leq \left(2\varepsilon \|u^\varepsilon\|^2_{L^\infty} - \frac{1}{2} \right) \int_{\mathbb{R}^2} |\nabla(u^\varepsilon - v)|^2 \, \mathrm{d}x + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon . \nabla v|^2 \, \mathrm{d}x \\
&\leq \left(\frac{1}{2C_1^2} - \frac{1}{2} \right) \int_{\mathbb{R}^2} |\nabla(u^\varepsilon - v)|^2 \, \mathrm{d}x + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon . \nabla v|^2 \, \mathrm{d}x \\
&\leq 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon . \nabla v|^2 \, \mathrm{d}x \ \leq \ 2\varepsilon \|u^\varepsilon\|^2_{L^\infty} \int_{\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x \ = \ \mathcal{O}\left(\varepsilon^s\right)
\end{aligned}$$

since we have (see (2.16) and assumptions (2.1))

$$\|u_0^\varepsilon\|_{L^\infty} \leq C_2 \|u_0^\varepsilon\|^\delta_{\dot{H}^\delta} \|u_0^\varepsilon\|^{1-\delta}_{\dot{H}^{1+\delta}} \lesssim \varepsilon^{\left(-\frac{\delta}{2} + \frac{s}{2}\right)\delta} \times \varepsilon^{\left(-\frac{1+\delta}{2} + \frac{s}{2}\right)(1-\delta)} = \varepsilon^{\frac{s-1}{2}}.$$

Then we have obtained the lemma, with $\mu = s$:

$$\tilde{A}^\varepsilon = \int_0^T A^\varepsilon \, \mathrm{d}t = \mathcal{O}(\varepsilon^s).$$

$\square$

2.2.2.4 Conclusion

Gathering the results in the previous three subsections, we obtain

$$E_{\varepsilon,v}(T) \leq 2c^2 \int_0^T \|v\|_{BMO}^2 E_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}\left(\varepsilon^{\nu}\right) + \mathcal{O}(\varepsilon^s) + E_{\varepsilon,v}(0).$$

Assuming that

$$\|u_0^\varepsilon - v_0\|_{L^2} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \tag{2.39}$$

in addition to assumptions (H) and (2.23), we have

$$E_{\varepsilon,v}(0) = \mathcal{O}\left(\varepsilon^s\right)$$

by the triangle inequality. It follows that

$$E_{\varepsilon,v}(T) \leq 2c^2 \int_0^T \|v_0\|_{BMO}^2 E_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}\left(\varepsilon^{\nu}\right) + \mathcal{O}\left(\varepsilon^s\right).$$

Now $\nu = \dfrac{s(1+\delta)-\tilde{\varepsilon}}{2(1+\delta-\tilde{\varepsilon})} \leq \dfrac{s}{2}$ since $s \leq 1$ so

$$E_{\varepsilon,v}(T) \leq 2 \int_0^T \|v_0\|_{BMO}^2 E_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}\left(\varepsilon^{\nu}\right).$$

Gronwall lemma (and $v \in L_T^2 BMO(\mathbb{R}^2)$) yields

$$E_{\varepsilon,v}(T) = \mathcal{O}\left(\varepsilon^{\nu}\right)$$

for all positive T and, using inequality (2.28)

$$\int_{\mathbb{R}^2} |u^\varepsilon - v|^2\,\mathrm{d}x \leq 4E_{\varepsilon,v}(t),$$

we deduce the convergence in the $L_T^\infty L^2(\mathbb{R}^2)$ norm of u^ε solution to (HNS^ε) towards v solution to the (NS) equations with initial data in $H^s(\mathbb{R}^2)^2$, where $0 < s \leq 1$.
Theorem 2.1 is now proved.

2.3 The 3D case : proof of Theorem 2.2

In this section, we shall follow the plan of the previous one : we shall start by showing global existence for the damped wave equation (HNS^ε) then we will prove the convergence of this global solution u^ε towards the solution to Navier-Stokes problem with initial data in $H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ and $0 < s < 1$. Let us recall again that the critical space for (NS) in $\mathbb{R}^3$ is $\dot{H}^{\frac{1}{2}}$.

2.3.1 Global existence in $\dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3, \mathbb{R}^3)$

We apply the same fixed point method as in the two-dimensional case. We perform the same scale change (2.5) and retrieve system (HNS) with initial data $(u_0, u_1) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$.
We conclude then that there exists a local solution u to (HNS), defined on the time interval $[0, T)$, for all positive real T satisfying

$$T \leq \frac{1}{12 + 72C\left(\|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}\right)} \cdot \tag{2.40}$$

In particular, while $\|u(t)\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}$ is bounded, we can reiterate the fixed point argument and extend the solution. Let $T^{\max}$ be the maximal existence time. We shall prove that $T^{\max} = +\infty$. Following the same reasoning by contradiction as in the previous section, we assume that $T^{\max} < +\infty$. This would imply that

$$\|u(t)\|_X := \|u(t)\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \longrightarrow +\infty$$

as t goes to $T^{\max}$.
Let us resume our wave equation (HNS^ε) with parameter ε. The scaling (2.5) gives

$$\|u\|_X = \varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} + \varepsilon\|\partial_t u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}}\right).$$

Let us define, for positive δ, the energy

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) = \int_{\mathbb{R}^3}\left(\frac{1}{2}|\Lambda^{\frac{1}{2}+\delta}(u^\varepsilon + \varepsilon\partial_t u^\varepsilon)|^2 + \frac{\varepsilon^2}{2}|\Lambda^{\frac{1}{2}+\delta}\partial_t u^\varepsilon|^2 + \varepsilon|\Lambda^{\frac{3}{2}+\delta}u^\varepsilon|^2\right) dx.$$

Then we have

$$\varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \varepsilon\|\partial_t u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}\right) \leq C\varepsilon^{\frac{\delta}{2}}\sqrt{E_\varepsilon^\delta}.$$

If we prove that $\varepsilon^{\frac{\delta}{2}}\sqrt{E_\varepsilon^\delta}$ is bounded, we can deduce that the solution u^ε is global.

Globalization Let $\delta > 0$ and consider the energy $E_\varepsilon^{\frac{1}{2}+\delta}$ defined as above. We shall prove that there exists a positive constant C_0 such that

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq C_0\varepsilon^{-\delta} \tag{2.41}$$

for all time $t > 0$. To this purpose, we shall adapt the method used in the two-dimensional case to three space dimensions. To get around the difficulties, we use mainly interpolations and product estimates.

First, let us recall that $\dot{H}^{\frac{3}{2}+\delta} \cap L^\infty(\mathbb{R}^3)$ is an algebra (see [1]) and that the inequality

$$\|fg\|_{\dot{H}^{\frac{3}{2}+\delta}} \leq C_1 \left(\|f\|_{L^\infty} \|g\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|g\|_{L^\infty} \|f\|_{\dot{H}^{\frac{3}{2}+\delta}} \right) \tag{2.42}$$

holds for all $f, g \in \dot{H}^{\frac{3}{2}+\delta} \cap L^\infty(\mathbb{R}^3)$. Moreover, using the embedding $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ and the interpolation inequality

$$\|f\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)} \leq C \|f\|^{\delta}_{\dot{H}^{\frac{1}{2}+\delta}} \|f\|^{1-\delta}_{\dot{H}^{\frac{3}{2}+\delta}},$$

we obtain

$$\|f\|_\infty \leq C_2 \|f\|^{\delta}_{\dot{H}^{\frac{1}{2}+\delta}} \|f\|^{1-\delta}_{\dot{H}^{\frac{3}{2}+\delta}}. \tag{2.43}$$

Remark. *In $\mathbb{R}^3$, the Navier-Stokes problem with initial data v_0 in $H^{\frac{1}{2}}$ has a global solution if $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough. So we could show that $E_\varepsilon^{\frac{1}{2}+\delta} + N E_\varepsilon^{\frac{1}{2}}$ decreases and deduce the globality of the solution but this method requires a smallness assumption on the $\dot{H}^{\frac{1}{2}}$ norm of u_0^ε which depends on δ and we would have*

$$\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \to 0$$

when δ goes to 0.

We shall prove that $E_\varepsilon^{\frac{1}{2}+\delta} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$ decreases if N is large enough and if the $\dot{H}^{\frac{1}{2}}$ norm of u^ε is small enough. First, we have to prove that $E_\varepsilon^{\frac{1}{2}}$ decreases. In this purpose, let us compute the time derivative of the energy.

$$\begin{aligned} \frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) &= -\int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac{1}{2}} u^\varepsilon \, \mathrm{d}x + \int_{\mathbb{R}^3} \left(\varepsilon |\Lambda^{\frac{1}{2}}(u^\varepsilon.\nabla u^\varepsilon)|^2 - |\Lambda^{\frac{3}{2}} u^\varepsilon|^2 \right) \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} u^\varepsilon.\nabla u^\varepsilon.\Lambda u^\varepsilon \, \mathrm{d}x + \left((6C_1)^2 \varepsilon \|u^\varepsilon\|^2_\infty - 1 \right) \|u^\varepsilon\|^2_{\dot{H}^{\frac{3}{2}}}. \end{aligned} \tag{2.44}$$

In order to estimate the first term on the right hand side, we state the following theorem :

Theorem 2.5. *Let $f \in H^{\frac{3}{2}}(\mathbb{R}^3)$. Then the following estimate holds.*

$$\left| \int_{\mathbb{R}^3} \Lambda f.(f.\nabla f) \, dx \right| \leq \|\Lambda^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{3}{2}} f\|^2_{L^2(\mathbb{R}^3)}. \tag{2.45}$$

Proof. Let $u \in H^{\frac{3}{2}}(\mathbb{R}^3)$. First, we use the duality between $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$.

$$|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)\,\mathrm{d}x| \leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}.$$

We shall estimate the $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ norm of $f.\nabla f$. For this sake, let us notice that $f \in H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ by a Sobolev embedding. Thus, by Hölder inequality, we have

$$f.\nabla f \in L^{\frac{3}{2}}(\mathbb{R}^3)$$

since $\frac{1}{6} + \frac{1}{2} = \frac{2}{3}$. Moreover, we shall use the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$ on its dual form $L^{\frac{3}{2}}(\mathbb{R}^3) \subset H^{-\frac{1}{2}}(\mathbb{R}^3)$. Finally, we obtain

$$\begin{aligned}
|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)\,\mathrm{d}x| &\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \\
&\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
&\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f\|_{L^6(\mathbb{R}^3)} \|\nabla f\|_{L^2(\mathbb{R}^3)} \\
&\leq \|f\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \|f\|^2_{\dot{H}^1(\mathbb{R}^3)}.
\end{aligned}$$

Now, we use the Gagliardo-Nirenberg inequality

$$\|f\|_{\dot{H}^1} \leq \sqrt{\|f\|_{\dot{H}^{\frac{3}{2}}} \|f\|_{\dot{H}^{\frac{1}{2}}}}\,.$$

Thus

$$|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)\,\mathrm{d}x| \leq \|f\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)},$$

which finishes the proof. □

Using theorem 2.5, we shall prove the following lemma.

Lemma 2.8. *Let $T_\varepsilon^{\max}$ be the maximal existence time for (HNS^ε) and define $0 \leq T \leq T_\varepsilon^{\max}$ by*

$$T = \sup\left\{0 \leq \tau \leq T_\varepsilon^{\max} \ : \ \forall\, t \in [0, \tau[,\ \|u^\varepsilon(t)\|_\infty < \frac{1}{7C_1\sqrt{\varepsilon}}\right\}.$$

Assume $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \dfrac{1}{16}$ and

$$\varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \sqrt{\varepsilon}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} = o(1) \tag{2.46}$$

when ε goes to 0. Then the energy $E_\varepsilon^{\frac{1}{2}}$ decreases on $[0,T)$ and we have

$$\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) \leq -\frac{1}{8}\|u^\varepsilon(t)\|^2_{\dot H^{\frac{3}{2}}} \tag{2.47}$$

for all $t \in [0,T)$.

Proof. Let $t \in [0,T]$. Since $\|u^\varepsilon(t)\|_\infty < \dfrac{1}{7C_1\sqrt{\varepsilon}}$, inequality (2.44) becomes

$$\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) \leq \int_{\mathbb{R}^3} u^\varepsilon.\nabla u^\varepsilon.\Lambda u^\varepsilon \, \mathrm{d}x - \frac{1}{4}\|u^\varepsilon\|^2_{\dot H^{\frac{3}{2}}}.$$

Now, theorem 2.5 (applied to first term on the right hand side) yields

$$\begin{aligned}
\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) &\leq \|u^\varepsilon(t)\|_{\dot H^{\frac{1}{2}}}\|u^\varepsilon(t)\|^2_{\dot H^{\frac{3}{2}}} - \frac{1}{4}\|u^\varepsilon(t)\|^2_{\dot H^{\frac{3}{2}}} \\
&\leq \left(\|u^\varepsilon(t)\|_{\dot H^{\frac{1}{2}}} - \frac{1}{4}\right)\|u^\varepsilon(t)\|^2_{\dot H^{\frac{3}{2}}}.
\end{aligned}$$

In $t=0$, this inequality gives that $E_\varepsilon^{\frac{1}{2}}$ decreases in a neighborhood of 0. Let

$$\tau = \sup\left\{0 \leq \tilde\tau \leq T \ : \ E_\varepsilon^{\frac{1}{2}} \text{ decreases on } [0,\tau]\right\}.$$

So $0 < \tau \leq T$. Assume that $\tau < T$. On the one hand, we have

$$\|u^\varepsilon(\tau)\|^2_{\dot H^{\frac{1}{2}}} < 2E_\varepsilon^{\frac{1}{2}}(\tau).$$

On the other hand, for ε small enough, we have

$$E_\varepsilon^{\frac{1}{2}}(0) \leq \|u_0^\varepsilon\|^2_{\dot H^{\frac{1}{2}}} + \frac{3\varepsilon^2}{2}\|u_1^\varepsilon\|^2_{\dot H^{\frac{1}{2}}} + \varepsilon\|u_0^\varepsilon\|^2_{\dot H^{\frac{3}{2}}} < 2\|u_0^\varepsilon\|^2_{\dot H^{\frac{1}{2}}}$$

due to assumptions (2.46). Adding to that the energy decay on $[0,\tau]$, we obtain

$$\|u^\varepsilon(\tau)\|^2_{\dot H^{\frac{1}{2}}} < 2E_\varepsilon^{\frac{1}{2}}(\tau) \leq 2E_\varepsilon^{\frac{1}{2}}(0) \leq 4\|u_0^\varepsilon\|^2_{\dot H^{\frac{1}{2}}} \tag{2.48}$$

so

$$\|u^\varepsilon(\tau)\|_{\dot H^{\frac{1}{2}}} < 2 \times \frac{1}{16} = \frac{1}{8}.$$

thus $E_\varepsilon^{\frac{1}{2}}$ decreases in a neighborhood of τ and this is a contradiction with its definition.
Therefore $E_\varepsilon^{\frac{1}{2}}$ decreases on $[0,T]$ and the inequality (2.47) is true for all $t \in [0,T]$. □

Lemma 2.9. *Assume* $\|u_0^\varepsilon\|_{\dot H^{\frac12}} < \dfrac{1}{16}$ *and*

$$(H_1) \qquad i)\ \varepsilon^{\frac{1+\delta}{2}} \|u_0^\varepsilon\|_{\dot H^{\frac32+\delta}} = o(1)\ , \qquad ii)\ \varepsilon^{\frac{\delta}{2}} \|u_0^\varepsilon\|_{\dot H^{\frac12+\delta}} = o(1).$$

when ε *goes to zero.*
Recall that $0 \le T \le T_\varepsilon^{\max}$ *is defined by*

$$T = \sup\left\{ 0 \le \tau \le T_\varepsilon^{\max}\ :\ \forall\ t \in [0,\tau[,\ \|u^\varepsilon(t)\|_\infty < \frac{1}{7C_1\sqrt{\varepsilon}} \right\}.$$

Then $T > 0$ *and there exists a large number* N*, depending only on* δ*, and a constant* $C > 1$ *such that*

$$E_\varepsilon^{\frac12+\delta}(t) \le C^N E_\varepsilon^{\frac12+\delta}(0)$$

for all $t \in [0,T)$ *and* ε *small enough.*

Proof. Let us compute the derivative of the energy.

$$\begin{aligned}
\frac{d}{dt} E_\varepsilon^{\frac12+\delta}(t) &= \int_{\mathbb{R}^3} \Big(\Lambda^{\frac12+\delta}(\varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon + \varepsilon\partial_t u^\varepsilon) + \\
&\quad + \varepsilon^2 \Lambda^{\frac12+\delta}\partial_t u^\varepsilon.\Lambda^{\frac12+\delta}\partial_{tt}u^\varepsilon + 2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon \Big)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \Big(\Lambda^{\frac12+\delta}(\varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon) + \\
&\quad + \Lambda^{\frac52+\delta}u^\varepsilon.\Lambda^{\frac12+\delta}\,(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon) - \varepsilon^2\Lambda^{\frac12+\delta}\partial_{tt}u^\varepsilon.\Lambda^{\frac12+\delta}\partial_t u^\varepsilon - \\
&\quad - \varepsilon|\Lambda^{\frac12+\delta}\partial_t u^\varepsilon|^2 + \varepsilon^2\Lambda^{\frac12+\delta}\partial_t u^\varepsilon.\Lambda^{\frac12+\delta}\partial_{tt}u^\varepsilon + \\
&\quad + 2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon \Big)\,\mathrm{d}x.
\end{aligned}$$

Now, recall that u^ε is a solution to (HNS^ε). Then

$$\begin{aligned}
\frac{d}{dt} E_\varepsilon^{\frac12+\delta}(t) &= \int_{\mathbb{R}^3} \Big(-\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon) - \\
&\quad - \Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\,(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon) - \varepsilon|\Lambda^{\frac12+\delta}\partial_t u^\varepsilon|^2 + \\
&\quad + 2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon \Big)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon\,\mathrm{d}x - \\
&\quad - \int_{\mathbb{R}^3} 2\varepsilon\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}\partial_t u^\varepsilon\,\mathrm{d}x - \\
&\quad - \int_{\mathbb{R}^3} \varepsilon|\Lambda^{\frac12+\delta}\partial_t u^\varepsilon|^2 - |\Lambda^{\frac32+\delta}u^\varepsilon|^2\,\mathrm{d}x.
\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}E_\varepsilon^{\frac12+\delta}(t) &\leq -\int_{\mathbb{R}^3}\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon\,\mathrm{d}x+\\ &\quad+\int_{\mathbb{R}^3}\left(\varepsilon|\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon)|^2-|\Lambda^{\frac32+\delta}u^\varepsilon|^2\right)\mathrm{d}x\\ &= I+II.\end{aligned}$$

First, let us estimate the integral I. Cauchy-Schwarz inequality followed by (2.42) yields

$$\begin{aligned}|I| &\leq \|u^\varepsilon.\nabla u^\varepsilon\|_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\ &\leq 3\times 2C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\ &\leq 6C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}.\end{aligned}\tag{2.49}$$

Thus, back to (2.49), we have

$$\begin{aligned}|I| &\leq 6C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\ &\leq 6C_1C_2\|u^\varepsilon\|^{1+\delta}_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}}\\ &\leq 6C_1C_2\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{\delta}_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}}\\ &\leq 6C_1C_2C_3\|u^\varepsilon\|^{1-\delta}_{\dot H^{\frac12}}\left(\|u^\varepsilon\|_{\dot H^{\frac32}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\right)^\delta\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}},\end{aligned}$$

where we have used another interpolation inequality between $\dot H^{\frac12}$ and $\dot H^{\frac32}$:

$$\|f\|_{\dot H^{\frac12+\delta}}\leq C_3\|f\|^{1-\delta}_{\dot H^{\frac12}}\|f\|^{\delta}_{\dot H^{\frac32}}.$$

Furthermore, the inequality

$$\begin{aligned}II &\leq \left((6C_1)^2\varepsilon\|u^\varepsilon\|^2_\infty-1\right)\int_{\mathbb{R}^3}|\Lambda^{\frac32+\delta}u^\varepsilon|^2\,\mathrm{d}x\\ &\leq -\frac14\|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}}\end{aligned}\tag{2.50}$$

holds if

$$\|u^\varepsilon(t)\|_\infty\leq\frac{1}{7C_1\sqrt{\varepsilon}}.\tag{2.51}$$

For ε small enough, this condition is ensured at time $t=0$ due to assumptions (H_1). By continuity of the local solution u^ε with respect to the time variable

t, we deduce that $T > 0$. Let $t < T$.
A Young inequality yields

$$\begin{aligned} \frac{d}{dt} E_\varepsilon^{\frac{1}{2}+\delta}(t) &\leq \delta 2^{2\frac{2-\delta}{\delta}} (6C_1C_2C_3)^{\frac{2}{\delta}} \|u^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^{2\frac{1-\delta}{\delta}} \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &\leq \delta 2^{2\frac{2-\delta}{\delta}} (6C_1C_2C_3)^{\frac{2}{\delta}} \left(4\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2\right)^{\frac{1-\delta}{\delta}} \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &< 16\delta\, (3C_1C_2C_3)^{\frac{2}{\delta}} \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 E_\varepsilon^{\frac{1}{2}+\delta}(t). \end{aligned}$$

To alleviate the notations, let us set $K = K(\delta, C_1, C_2, C_3) = 16\delta\, (3C_1C_2C_3)^{\frac{2}{\delta}}$.

As above, in subsection 2.2.1.2, we prove that $\mathcal{E}_{\delta,N} := E_\varepsilon^{\frac{1}{2}+\delta}\left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$ decreases on $[0, T)$ for ε small enough and N large enough.
Let $0 \leq t < T$ and $N \geq 0$.
We have

$$\begin{aligned} \frac{d\mathcal{E}_{\delta,N}}{dt}(t) &= N \frac{dE_\varepsilon^{\frac{1}{2}}}{dt} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta} + \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N \frac{dE_\varepsilon^{\frac{1}{2}+\delta}}{dt}(t) \\ &\leq \left[-\frac{N}{8} + K\left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)\right] \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &\leq \left[-\frac{N}{8} + K\left(1 + 2\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2\right)\right] \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t). \end{aligned}$$

Inequality (2.48) yields

$$\frac{d\mathcal{E}_{\delta,N}}{dt}(t) \leq \left(-\frac{N}{8} + \frac{129}{128}K\right) \|u^\varepsilon(t)\|_{\dot{H}^{\frac{3}{2}}}^2 \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t).$$

Now, taking $N \geq 8K \times \frac{129}{128} = 129\delta\, (3C_1C_2C_3)^{\frac{2}{\delta}}$, we obtain that

$$\mathcal{E}_{\delta,N} = E_\varepsilon^{\frac{1}{2}+\delta}\left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$$

decays on $[0, T)$. So we have

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq E_\varepsilon^{\frac{1}{2}+\delta}(0)\left(1 + E_\varepsilon^{\frac{1}{2}}(0)\right)^N \leq E_\varepsilon^{\frac{1}{2}+\delta}(0)\left(\frac{129}{128}\right)^N$$

for $t \in [0, T)$. $\square$

Lemma 2.10. *Assume the limit*

$$\varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} \longrightarrow 0\ ,\quad \varepsilon \to 0 \tag{2.52}$$

in addition to the assumptions (H_1) *and* (2.46)*. Then we have*

$$E_\varepsilon^{\frac{1}{2}+\delta}(0) = o\left(\varepsilon^{-\delta}\right).$$

Moreover, for all $t \in [0,T)$ *and* ε *small enough, the inequality*

$$\|u^\varepsilon\|_\infty \ \leq\ \frac{1}{14C_1\sqrt{\varepsilon}}$$

holds.

We skip the proof here as it is analogous to the one of Lemma 2.2. As above, this lemma implies the inequality (2.41) and the globality of the solution u^ε.

2.3.2 Convergence towards a solution to (NS) problem

The aim of this subsection is to prove an error estimate in the $L^\infty_T \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ norm. More precisely, let $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$, $0 < s < 1$, be the initial data for the Navier-Stokes equations (NS) and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta} \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$, $0 < \delta < 1$, be the initial data for the damped wave equation (HNS^ε). We will prove that, if $\|u_0^\varepsilon - v_0\|^2_{\dot{H}^{\frac{1}{2}}} = \mathcal{O}(\varepsilon^s)$, then for all positive T, we prove that

$$\sup_{t\in(0,T)} \|(u^\varepsilon - v)(t)\|^2_{\dot{H}^{\frac{1}{2}}} = \mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right).$$

In the following, we adapt the method in section 2.2.2. So let us consider again the Dafermos modulated energy defined by

$$\mathcal{E}_{\varepsilon,v}(t) = \frac{1}{2}\|u^\varepsilon - v + \varepsilon\partial_t u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{\varepsilon^2}{2}\|\partial_t u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} + \varepsilon\|\nabla u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} \tag{2.53}$$

for all divergence-free vector field $v(t,x)$. As above, this energy satisfies the inequality

$$\|u^\varepsilon(t) - v(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq 4\mathcal{E}_{\varepsilon,v}(t). \tag{2.54}$$

We shall show that the assumptions (2.3) in Theorem 2.2 and a Gronwall inequality imply that

$$\forall\ t > 0\ ,\quad \mathcal{E}_{\varepsilon,v}(t) = \mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right).$$

Let us compute the derivative of $\mathcal{E}_{\varepsilon,v}$.

Lemma 2.11. *The Dafermos modulated energy $\mathcal{E}_{\varepsilon,v}$ satisfies the identity*

$$\begin{aligned}
\frac{d\mathcal{E}_{\varepsilon,v}}{dt} &= -\varepsilon\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}}\partial_t v.\Lambda^{\frac{1}{2}}\partial_t u^\varepsilon\,dx+\int_{\mathbb{R}^3}\Lambda(u^\varepsilon-v).\mathbb{P}\nabla:(u^\varepsilon\otimes u^\varepsilon-v\otimes v)\,dx\\
&+\int_{\mathbb{R}^3}\left(\varepsilon|\Lambda^{\frac{1}{2}}\mathbb{P}\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2-|\Lambda^{\frac{3}{2}}(u^\varepsilon-v)|^2\right)\,dx\\
&-\varepsilon\int_{\mathbb{R}^3}|\Lambda^{\frac{1}{2}}\left(\partial_t u^\varepsilon+\mathbb{P}\nabla:u^\varepsilon\otimes u^\varepsilon\right)|^2\,dx\\
&-\int_{\mathbb{R}^3}\Lambda(\partial_t v+\mathbb{P}\nabla v\otimes v-\Delta v).(v-u^\varepsilon)\,dx. \qquad (2.55)
\end{aligned}$$

As above in section 2.2.2, we take v solution to $\partial_t v+\mathbb{P}\nabla:v\otimes v-\Delta v=0$ in the following, so that the last term in identity (2.55) vanishes. Moreover, the penultimate term in (2.55) being negative, we have

$$\begin{aligned}
\frac{d\mathcal{E}_{\varepsilon,v}}{dt} &\leq -\varepsilon\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}}\partial_t v.\Lambda^{\frac{1}{2}}\partial_t u^\varepsilon\,\mathrm{d}x+\int_{\mathbb{R}^3}\Lambda(u^\varepsilon-v).\mathbb{P}\nabla:(u^\varepsilon\otimes u^\varepsilon-v\otimes v)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\left(\varepsilon|\Lambda^{\frac{1}{2}}\mathbb{P}\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2-|\Lambda^{\frac{3}{2}}(u^\varepsilon-v)|^2\right)\,\mathrm{d}x. \qquad (2.56)
\end{aligned}$$

Concerning the first term, the argument in subsection 2.2.2.1 applies and immediately yields the estimate

$$\varepsilon\int_0^T\!\!\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}}\partial_t u^\varepsilon.\Lambda^{\frac{1}{2}}\partial_t v\,\mathrm{d}t\,\mathrm{d}x=\mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right).$$

We shall estimate the two remaining terms on the right hand side in the following two subsections.

2.3.2.1 Estimating $\int_0^T\!\int_{\mathbb{R}^3}\Lambda(u^\varepsilon-v).\nabla:(u^\varepsilon\otimes u^\varepsilon-v\otimes v)\,\mathbf{d}t\,\mathbf{d}x$

First, let us recall the identity

$$\nabla:(u^\varepsilon\otimes u^\varepsilon-v\otimes v)=(v-u^\varepsilon).\nabla v+u^\varepsilon.\nabla(v-u^\varepsilon)$$

and let us estimate the integral

$$I(t):=-\int_{\mathbb{R}^3}\Lambda(u^\varepsilon-v).\nabla:(u^\varepsilon\otimes u^\varepsilon-v\otimes v)\,\mathrm{d}x$$

as follows. Using a variant of theorem 2.5 above, we obtain

$$\begin{aligned}
I(t) &= \int_{\mathbb{R}^3}\Lambda(u^\varepsilon-v).\left((v-u^\varepsilon).\nabla v+u^\varepsilon.\nabla(v-u^\varepsilon)\right)\,\mathrm{d}x\\
&\lesssim \|u^\varepsilon-v\|_{\dot H^{\frac{3}{2}}}\left(\|u^\varepsilon-v\|_{L^6}\|v\|_{\dot H^1}+\|u^\varepsilon\|_{L^6}\|u^\varepsilon-v\|_{\dot H^1}\right)\\
&\lesssim \|u^\varepsilon-v\|_{\dot H^{\frac{3}{2}}}\|u^\varepsilon-v\|_{\dot H^1}\left(\|v\|_{\dot H^1}+\|u^\varepsilon\|_{\dot H^1}\right).
\end{aligned}$$

Now, a Galgliardo-Nirenberg inequality followed by a Young inequality yields

$$\begin{aligned}
|I(t)| &\leq C\|u^\varepsilon - v\|^{\frac{3}{2}}_{\dot H^{\frac{3}{2}}}\|u^\varepsilon - v\|^{\frac{1}{2}}_{\dot H^{\frac{1}{2}}}\left(\|v\|_{\dot H^1} + \|u^\varepsilon\|_{\dot H^1}\right) \\
&\leq \frac{C^4}{4}\|u^\varepsilon - v\|^2_{\dot H^{\frac{1}{2}}}\left(\|v\|_{\dot H^1} + \|u^\varepsilon\|_{\dot H^1}\right)^4 + \frac{3}{4}\|u^\varepsilon - v\|^2_{\dot H^{\frac{3}{2}}} \\
&\leq 2C^4\|u^\varepsilon - v\|^2_{\dot H^{\frac{1}{2}}}\left(\|v\|^4_{\dot H^1} + \|u^\varepsilon\|^4_{\dot H^1}\right) + \frac{3}{4}\|u^\varepsilon - v\|^2_{\dot H^{\frac{3}{2}}}.
\end{aligned}$$

Finally, using inequality (2.54), we obtain

$$|I(t)| \leq 8C^4\left(\|v\|^4_{\dot H^1} + \|u^\varepsilon\|^4_{\dot H^1}\right)\mathcal{E}_{\varepsilon,v}(t) + \frac{3}{4}\|u^\varepsilon - v\|^2_{\dot H^{\frac{3}{2}}}.$$

Now, let $T > 0$ and recall that v is a solution to the Navier-Stokes equations, with initial data $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ and $0 < s < 1$. So $v \in L^2_T H^{s+\frac{3}{2}} \cap L^\infty_T H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$. By interpolation, we have also $v \in L^4_T H^{s+1}(\mathbb{R}^3)^3$.
Furthermore, notice that inequality (2.48) implies that $u^\varepsilon \in L^\infty_T \dot H^{\frac{1}{2}}$ uniformly in ε and, using the energy decay proven in lemma 2.8, one can easily show that $u^\varepsilon \in L^2_T \dot H^{\frac{3}{2}}$ uniformly in ε. Thus, by interpolation, we obtain that

$$u^\varepsilon \in L^4_T \dot H^1(\mathbb{R}^3)^3$$

uniformly in ε. Therefore

$$\int_0^T I(t)\,\mathrm{d}t \leq 8C^4\int_0^T\left(\|v(t)\|^4_{H^{s+1}} + \|u^\varepsilon(t)\|^4_{\dot H^1}\right)\mathcal{E}_{\varepsilon,v}(t)\,\mathrm{d}t + \frac{3}{4}\int_0^T\|\nabla(u^\varepsilon - v)\|^2_{\dot H^{\frac{1}{2}}}\,\mathrm{d}t.$$

At this level, we have the following estimate :

$$\begin{aligned}
\mathcal{E}_{\varepsilon,v}(T) - \mathcal{E}_{\varepsilon,v}(0) \;\leq\;& 8C^4\int_0^T\left(\|v\|^4_{H^{s+1}} + \|u^\varepsilon\|^4_{\dot H^1}\right)\mathcal{E}_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right) \\
&+ \int_0^T\left(\varepsilon\|\nabla : u^\varepsilon \otimes u^\varepsilon\|^2_{\dot H^{\frac{1}{2}}} - \frac{1}{4}\|\nabla(u^\varepsilon - v)\|^2_{\dot H^{\frac{1}{2}}}\right)(t)\,\mathrm{d}t.
\end{aligned}$$

Let us set

$$\int_0^T A^\varepsilon(t)\,\mathrm{d}t := \int_0^T\left(\varepsilon\|\nabla : u^\varepsilon \otimes u^\varepsilon\|^2_{\dot H^{\frac{1}{2}}} - \frac{1}{4}\|\nabla(u^\varepsilon - v)\|^2_{\dot H^{\frac{1}{2}}}\right)(t)\,\mathrm{d}t.$$

It remains to estimate the last term $\int_0^T A^\varepsilon(t)\,\mathrm{d}t$.

2.3.2.2 Estimating $\int_0^T A^\varepsilon(t)\,\mathrm{d}t$

First, let us recall that $L^\infty \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ is an algebra. So

$$\|\nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u^\varepsilon \otimes u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} \leq C\|u^\varepsilon\|_\infty \|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}}.$$

Therefore we have

$$\begin{aligned} A^\varepsilon(t) &\leq C^2\varepsilon\|u^\varepsilon\|_\infty^2\|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}}^2 - \frac{1}{4}\|u^\varepsilon - v\|_{\dot{H}^{\frac{3}{2}}}^2 \\ &\leq \left(2C^2\varepsilon\|u^\varepsilon\|_\infty^2 - \frac{1}{4}\right)\|u^\varepsilon - v\|_{\dot{H}^{\frac{3}{2}}}^2 + 2C^2\varepsilon\|u^\varepsilon\|_\infty^2\|v\|_{\dot{H}^{\frac{3}{2}}}^2 \end{aligned}$$

by a Young inequality. Now, using that $\varepsilon\|u^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^3)}^2 = \mathcal{O}(\varepsilon^s)$ and that the solution v to (NS) is in $L_T^2\dot{H}^{\frac{3}{2}}$, we obtain

$$\int_0^T A^\varepsilon(t)\,\mathrm{d}t \leq 2C\varepsilon\|u^\varepsilon\|_{L^\infty}^2\|v\|_{L_T^2\dot{H}^{\frac{3}{2}}}^2 = \mathcal{O}(\varepsilon^s)$$

if ε is small enough.

2.3.2.3 Conclusion

Finally, we have

$$\mathcal{E}_{\varepsilon,v}(T) \leq 8C^4\int_0^T \left(\|v\|_{H^{s+1}}^4 + \|u^\varepsilon\|_{\dot{H}^1}^4\right)\mathcal{E}_{\varepsilon,v}(t)\,\mathrm{d}t + \mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right) + \mathcal{E}_{\varepsilon,v}(0).$$

Assuming that

$$\|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \sqrt{\varepsilon}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right),$$

we have immediately

$$\mathcal{E}_{\varepsilon,v}(0) = \mathcal{O}(\varepsilon^s).$$

Let us recall that $v \in L_T^4H^{s+1}(\mathbb{R}^3)^3$ and $u^\varepsilon \in L_T^4\dot{H}^1$. The Gronwall's lemma implies that

$$\mathcal{E}_{\varepsilon,v}(T) = \mathcal{O}\left(\varepsilon^{\left(\frac{s}{2}\right)^-}\right).$$

Now, (2.54) yields the announced error estimate and the proof of Theorem 2.2 is complete.

Chapitre 3

A finite propagation speed approximation

In this part, we introduce a finite propagation speed perturbation of the incompressible Navier-Stokes equations (NS). The model we consider is inspired by a hyperbolic perturbation of the heat equation due to Cattaneo in [9, 10] and by an equation that Višik and Fursikov investigated in [40] in order to find statistical solutions to (NS). We prove that the solutions to the perturbed Navier-Stokes equations approximate those to (NS).

We use refined energy methods involving fractional Sobolev spaces and precise estimates on the nonlinear term due to the dyadic Littlewood-Paley decomposition.

3.1 Introduction

The purpose of this chapter is to approximate the solutions to the incompressible Navier-Stokes equations with quasi-critical regularity initial datum by solutions to a nonlinear wave equation with a finite speed of propagation which is obtained by penalizing the incompressibility constraint. First, let us recall the Navier-Stokes system which describes the motion of an incompressible, viscous and homogeneous newtonian fluid whose velocity and pressure are denoted by v and p respectively.

$$(NS) \quad \partial_t v(t,x) - \nu\Delta v(t,x) + (v.\nabla)v(t,x) = -\nabla p(t,x) \ , \quad \operatorname{div} v(t,x) = 0,$$

where $t > 0$ and $x \in \mathbb{R}^n$ for $n = 2, 3$. The velocity is a $\mathbb{R}^n$ valued vector field and the pressure is scalar. The coefficient of the Laplacian is the viscosity and, without loss of generality, is assumed to be 1 in the following.

Applying the Leray projector $\mathbb{P}$ which maps L^2 into $L^2_\sigma := \{u \in L^2 : \operatorname{div} u = 0\}$ to (NS), we obtain the equations

$$(NS) \qquad \partial_t v - \Delta v + \mathbb{P}(v.\nabla)v = 0 \ , \quad \operatorname{div} v = 0$$

from which we can recover the pressure p.
A first hyperbolic perturbation of (NS) has been obtained after relaxation of the Euler equations and rescaling variables (see [7] and references therein) :

$$(HNS^\varepsilon) \quad \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon + \mathbb{P}(u^\varepsilon.\nabla)u^\varepsilon = 0 \ , \quad \operatorname{div} u^\varepsilon = 0.$$

In [9], Cattaneo introduced this equation (without the nonlinear term) as a perturbation of the heat equation. In [7] and [36], the authors approximate the solutions to (NS) by solutions to (HNS^ε) under some assumptions on the size and the regularity of the initial data. In [20], we improve the results of [7] and [36]. Under weaker assumptions on the initial data size, we prove the convergence of solutions to (HNS^ε) towards solutions to (NS) in the critical Sobolev space norm, that is $\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$, as ε goes to 0. More precisely, the theorem is :

Theorem 3.1. *Let $n = 2$ or 3 and $0 < s, \delta < 1$. Let $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ be a sequence of initial data for problem (HNS^ε). Assume*

$$\begin{cases} \|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1}} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}}} &= \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1+\delta}} &= \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{n}{2}-1+\delta}} &= o(1)\,. \end{cases}$$

Moreover, if $n = 3$, assume that $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3} < \frac{1}{16}$.
Then, for ε small enough, there exists a global solution u^ε to system (HNS^ε) that converges, when ε goes to 0, in the $L^\infty_{loc}(\mathbb{R}^+; \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n)$ norm, towards the unique solution v to (NS), with v_0 as initial datum. Moreover, for all positive T, there exists a constant C_T, depending only on T and v, such that

$$\sup_{t\in[0,T]} \|u^\varepsilon - v\|^2_{\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)^n}\, dx \leq C_T \varepsilon^{\left(\frac{s}{2}\right)^-}.$$

Remark. *In the 3D case, as a consequence of the smallness assumptions $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \frac{1}{16}$ and $\|u_0^\varepsilon - v_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we obtain the smallness of $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$, which is a necessary condition to the existence of a global strong solution to the Navier-Stokes equations in $\mathbb{R}^3$.*

In this paper, we shall consider another hyperbolic perturbation of (NS) which is inspired by Cattaneo's perturbation (HNS^{ε}) and by Višik and Fursikov's weakly compressible equations in [40]. The point is that we try to get around the difficulties which come from the Leray projector $\mathbb{P}$ (or, equivalently, from the pressure) on the one hand and, on the other hand, from the infinite propagation speed due to the heat kernel. The equation we introduce in this paper is :

$$(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha}).$$

Under the same assumptions as in Theorem 3.1, we prove that the solutions to $(HNS^{\varepsilon,\alpha})$ approximate those to (NS). Notice that we do not need any more restrictions involving α on the initial data.

Theorem 3.2. *Let $n = 2$ or 3 and $0 < s, \delta < 1$. Let $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ be a divergence-free vector field and $(u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha}) = (u_0^{\varepsilon}, u_1^{\varepsilon}) \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)^n \times H^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ be a sequence of divergence-free initial data for problem $(HNS^{\varepsilon,\alpha})$, independent of α. Assume*

$$\left\{\begin{array}{rcl} \|u_0^{\varepsilon,\alpha} - v_0\|_{\dot H^{\frac{n}{2}-1}} + \varepsilon\|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1}} + \varepsilon^{\frac{1}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1+\delta}} & = & \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}-1+\delta}} & = & o(1)\,. \end{array}\right. \tag{3.1}$$

Moreover, if $n = 3$, assume that $\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)^3} < \frac{1}{36K_2^3}$, where K_2 is the constant such that

$$\|f\|_{L^3(\mathbb{R}^3)} \leq K_2\|f\|_{\dot H^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Then, for ε, α small enough and for all positive T, the global solutions to $(HNS^{\varepsilon,\alpha})$ approximate those to (NS) in the $L^\infty_T \dot H^{\frac{n}{2}-1}(\mathbb{R}^n)^n$ norm.

Remark. *A possible choice of initial data is $\widehat{u_0^{\varepsilon,\alpha}}(\xi) = \widehat{v_0}(\xi)\mathbf{1}_{\{\sqrt{\varepsilon}|\xi|<1\}}$ and $u_1^{\varepsilon,\alpha} = 0$.*

In this paper, we shall prove that the solutions to $(HNS^{\varepsilon,\alpha})$ with initial data $(u_0^{\varepsilon}, u_1^{\varepsilon})$ converge, as α goes to 0, towards the solutions to (HNS^{ε}) with the same initial data. Then we conclude according to Theorem 3.1.

The next section is dedicated to the introduction of the model. In section 3.3, we shall prove that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation using the results of section 3.4 where we prove local existence. Then,we focus on the 2D case in section 3.5 : first, we recall in subsection 3.5.1 some important estimates on the solutions u^ε to (HNS^ε) then we prove global existence for $(HNS^{\varepsilon,\alpha})$ in part 3.5.2 and we show that its solutions approximate those to (HNS^ε) in subsection 3.5.3. Finally, section 3.6 is devoted to the 3D case and follows the same plan as section 3.5 : in subsection 3.6.1 we recall the useful regularity results on u^ε then we prove that the local solutions to $(HNS^{\varepsilon,\alpha})$ are global in subsection 3.6.2 and that the global solutions approximate those to (HNS^ε) in subsection 3.6.3 . Finally, Theorem 3.1 allows to conclude the proof of Theorem 3.2. Some important estimates coming from the framework of Littlewood-Paley theory are recalled in appendix B.

3.2 Introducing the model

In this section we shall introduce the finite speed of propagation equation that we will study in the next sections. We will work in the setting of $\mathbb{R}^2$. The 3D case is similar.
First, let us perturb the Navier-Stokes system (NS) into the damped nonlinear wave equation (HNS^ε) which we recall :

$$(HNS^\varepsilon) \quad \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon = -(u^\varepsilon.\nabla)u^\varepsilon - \nabla p^\varepsilon \quad , \quad \operatorname{div} u^\varepsilon = 0.$$

Then, applying the div $= \sum_{i=1}^n \partial_i(.)_i$ operator to this equation , we obtain

$$0 = \operatorname{div} f(u^\varepsilon) - \Delta p^\varepsilon, \tag{3.2}$$

where $f(u) = -(u.\nabla)u$. Let us now consider the stationary problem

$$-\Delta w^\varepsilon = f - \nabla p^\varepsilon \quad , \quad \operatorname{div} w^\varepsilon = 0 \tag{3.3}$$

which is related to (3.2). Indeed, applying div to (3.3), we obtain (3.2). Given $f \in \dot H^{-1}$, we look for a solution $w^\varepsilon \in \dot H^1$ to (3.3), *i.e.* such that

$$J^\varepsilon(w^\varepsilon) = \min\left\{ J^\varepsilon(v) \ : \ v \in \dot H^1, \operatorname{div} v = 0\right\},$$

where $J^\varepsilon(v) = \int \left(\frac{1}{2}|\nabla v|^2 - f.v\right) dx$. Now, using a penalization method, we change the problem into minimizing

$$J^{\varepsilon,\alpha}(v) = J^\varepsilon(v) + \frac{1}{2\alpha}\int |\operatorname{div} v|^2\, dx$$

in the space $\dot H^1$ so that the constraint $\operatorname{div} w^\varepsilon = 0$ is integrated to the functional $J^{\varepsilon,\alpha}$ to minimize. Let us call the minimizer $w^{\varepsilon,\alpha}$. Letting α go to zero in

$w^{\varepsilon,\alpha}$, we obtain the desired solution w^ε. Since $w^{\varepsilon,\alpha}$ minimizes $J^{\varepsilon,\alpha}$, we know that it solves

$$-\Delta w^{\varepsilon,\alpha} = f + \frac{1}{\alpha}\nabla(\operatorname{div} w^{\varepsilon,\alpha}) = 0. \tag{3.4}$$

Now, recall that $f \in \dot{H}^{-1}$ and write

$$\begin{aligned}
\int_{\mathbb{R}^2}\left(|\nabla w^{\varepsilon,\alpha}|^2 + \frac{1}{\alpha}|\operatorname{div} w^{\varepsilon,\alpha}|^2\right)dx &= \int_{\mathbb{R}^2} f.w^{\varepsilon,\alpha}\,dx \\
&\leq \|f\|_{\dot{H}^{-1}}\|w^{\varepsilon,\alpha}\|_{\dot{H}^1} \\
&\leq \frac{1}{2}\|f\|^2_{\dot{H}^{-1}} + \frac{1}{2}\|w^{\varepsilon,\alpha}\|^2_{\dot{H}^1}.
\end{aligned}$$

So we immediately deduce that

$$\|\operatorname{div} w^{\varepsilon,\alpha}\|^2_{L^2} \leq C\alpha\|f\|^2_{\dot{H}^{-1}} = \mathcal{O}(\alpha). \tag{3.5}$$

On this basis, we shall consider the equation

$$(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha}).$$

Due to (3.5), we say that $(HNS^{\varepsilon,\alpha})$ is weakly compressible. Let us point out here that this model reminds of the one studied by Višik and Fursikov in [40] and, later, by Basson [2] and Lelièvre [27] in order to study statistical solutions to the Navier-Stokes equations.

In the next section, we shall prove that $(HNS^{\varepsilon,\alpha})$ has a finite propagation speed through Picard's fixed point theorem.

3.3 Finite speed of propagation

In order to prove that the equation

$$(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha})$$

has a finite propagation speed, let us consider the Helmholtz-Hodge decomposition of $u^{\varepsilon,\alpha}$:

$$u^{\varepsilon,\alpha} = w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha},$$

where $w^{\varepsilon,\alpha} = \mathbb{Q}u^{\varepsilon,\alpha} := \frac{1}{\Delta}\nabla\operatorname{div} u^{\varepsilon,\alpha}$ is irrotational and $z^{\varepsilon,\alpha} = \mathbb{P}u^{\varepsilon,\alpha} := u^{\varepsilon,\alpha} - w^{\varepsilon,\alpha}$ is divergence-free. We obtain the system

$$\left\{\begin{array}{rcl}
\varepsilon\partial_{tt}z^{\varepsilon,\alpha} + \partial_t z^{\varepsilon,\alpha} - \Delta z^{\varepsilon,\alpha} &=& -\mathbb{P}\left((w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha}).\nabla\right)(w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha}) \\
\varepsilon\partial_{tt}w^{\varepsilon,\alpha} + \partial_t w^{\varepsilon,\alpha} - \frac{\alpha+1}{\alpha}\Delta w^{\varepsilon,\alpha} &=& -\mathbb{Q}\left((w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha}).\nabla\right)(w^{\varepsilon,\alpha} + z^{\varepsilon,\alpha})
\end{array}\right.$$

from which we can deduce a Duhamel's formula for the initial value problem

$$(P) \quad \begin{cases} (HNS^{\varepsilon,\alpha}) \\ u|_{t=0} = u_0 \ , \quad \partial_t u|_{t=0} = u_1 \end{cases} .$$

In section 3.4, we show local existence through Picard's contraction theorem in a small ball of the complete metric space

$$X_T \quad = \quad \left\{ (u, \partial_t u) \in \dot{H}^{\frac{n}{2}+\delta} \cap \dot{H}^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \times \dot{H}^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \right\}.$$

The contractive map argument is detailed in section 3.4. In the following, we shall denote $u^{\varepsilon,\alpha}$ by u to alleviate the notations. Picard's fixed point theorem gives a sequence

$$(u^j, \partial_t u^j) \xrightarrow{X_T} (u, \partial_t u)$$

defined by

$$\varepsilon \partial_{tt} u^{j+1} - \Delta u^{j+1} = -\partial_t u^j - (u^j.\nabla) u^j + \frac{1}{\alpha} \nabla \mathrm{div}\, u^j. \tag{3.6}$$

Now, set $\tilde{u} = \mathrm{div}\, u$ and apply the div operator to $(HNS^{\varepsilon,\alpha})$. Doing so, we obtain the following system :

$$(S) \quad \begin{cases} \varepsilon \partial_{tt} \tilde{u} - \frac{\alpha+1}{\alpha} \Delta \tilde{u} & = & -\partial_t \tilde{u} - \mathrm{div}\,(u.\nabla) u \\ \varepsilon \partial_{tt} u - \quad\quad \Delta u & = & -\partial_t u - (u.\nabla) u + \frac{1}{\alpha} \nabla \tilde{u} \end{cases} .$$

Then the initial value problem (P) is equivalent to

$$(P') \quad \begin{cases} (S) \\ u|_{t=0} = u_0 \ , \quad \partial_t u|_{t=0} = u_1 \ , \quad \tilde{u}|_{t=0} = \mathrm{div}\, u_0 \ , \quad \partial_t \tilde{u}|_{t=0} = \mathrm{div}\, u_1 \end{cases} .$$

Remark. *As we shall see in the following, this way of writing the equation $(HNS^{\varepsilon,\alpha})$ is convenient for the proof of finite speed of propagation but, unless we assume that u and $\tilde{u}$ are smooth, we cannot prove directly that the Duhamel's formula related to system (S) is locally contractive (due to the term $\nabla \tilde{u}$). Besides, we cannot prove that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation through the Helmholtz-Hodge decomposition since the operators $\mathbb{P}$ and $\mathbb{Q}$ are non-local.*

Moreover, applying div to (3.6), we obtain the equation

$$\varepsilon \partial_{tt} \tilde{u}^{j+1} - \frac{\alpha+1}{\alpha} \Delta \tilde{u}^{j+1} = -\partial_t \tilde{u}^j - \mathrm{div}\,(u^j.\nabla) u^j.$$

Then we have the coupled system of equations

$$(S^j) \quad \begin{cases} \varepsilon \partial_{tt} \tilde{u}^{j+1} - \frac{\alpha+1}{\alpha} \Delta \tilde{u}^{j+1} & = & -\partial_t \tilde{u}^j - \mathrm{div}\,(u^j.\nabla) u^j \\ \varepsilon \partial_{tt} u^{j+1} - \quad\quad \Delta u^{j+1} & = & -\partial_t u^j - (u^j.\nabla) u^j + \frac{1}{\alpha} \nabla \tilde{u}^j \end{cases} .$$

Now, let us consider the problem

$$(P^j)\quad \begin{cases} (S^j) \\ u^{j+1}|_{t=0} = u_0 \quad , \quad \partial_t u^{j+1}|_{t=0} = u_1 \\ \tilde{u}^{j+1}|_{t=0} = \operatorname{div} u_0 \ , \quad \partial_t \tilde{u}^{j+1}|_{t=0} = \operatorname{div} u_1 \\ u^0 \ , \quad v^0 \equiv 0 \end{cases} .$$

From now on, we assume u_0 and u_1 to be supported in a ball $B(0,R)$ and we set

$$c_1 = \sqrt{\frac{\alpha+1}{\alpha\varepsilon}} \ , \quad c_2 = \frac{1}{\sqrt{\varepsilon}}.$$

Let us recall the standard energies associated to the wave equations in (S) :

$$E_{c_1}(\tilde{u})(t) = \int_{\Gamma_{c_1,t,x}} e_{c_1}(\tilde{u})(t,y)\,\mathrm{d}y \ , \quad E_{c_2}(u)(t) = \int_{\Gamma_{c_2,t,x}} e_{c_2}(u)(t,y)\,\mathrm{d}y,$$

where the densities are

$$\begin{aligned} e_{c_1}(\tilde{u})(t,y) &= \frac{\varepsilon}{2}\,|\partial_t \tilde{u}(t,y)|^2 + \frac{\alpha+1}{2\alpha}\,|\nabla \tilde{u}(t,y)|^2, \\ e_{c_2}(u)(t,y) &= \frac{\varepsilon}{2}\,|\partial_t u(t,y)|^2 + \frac{1}{2}\,|\nabla u(t,y)|^2 \end{aligned}$$

and the cones $\Gamma_{c_i,t,x}$ are defined by

$$\Gamma_{c_i,t,x} = \left\{ (s,y) \ : \ 0 \le s \le \frac{t}{c_i}, \ |y-x| \le t - c_i s \right\}.$$

Notice that, since $c_1 > c_2$, we have $\Gamma_{c_1,t,x} \subset \Gamma_{c_2,t,x}$. Now, assume that we know that $u^j,\ \partial_t u^j,\ \tilde{u}^j,\ \partial_t \tilde{u}^j = 0$ on the cone $\Gamma_{c_1,t,x}$. Then, multiplying the first equation in (S^j) by $\partial_t \tilde{u}^{j+1}$ and integrating on $\Gamma_{c_1,t,x}$, we obtain :

$$\begin{aligned} 0 &= \int_{\Gamma_{c_1,t,x}} \left(\varepsilon \partial_{tt} \tilde{u}^{j+1} + \partial_t \tilde{u}^j - \frac{\alpha+1}{\alpha} \Delta \tilde{u}^{j+1} + \operatorname{div}(u^j.\nabla) u^j \right) \partial_t \tilde{u}^{j+1}\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{\Gamma_{c_1,t,x}} \left(\varepsilon \partial_{tt} \tilde{u}^{j+1} - \frac{\alpha+1}{\alpha} \Delta \tilde{u}^{j+1} \right) \partial_t \tilde{u}^{j+1}\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{\Gamma_{c_1,t,x}} \frac{\varepsilon}{2} \partial_t |\partial_t \tilde{u}^{j+1}|^2 - \frac{\alpha+1}{\alpha} \sum_{i=1}^n \partial_{ii} \tilde{u}^{j+1}.\partial_t \tilde{u}^{j+1}\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{\Gamma_{c_1,t,x}} \frac{\varepsilon}{2} \partial_t |\partial_t \tilde{u}^{j+1}|^2 - \frac{\alpha+1}{\alpha} \sum_{i=1}^n \left[\partial_i \left(\partial_i \tilde{u}^{j+1}.\partial_t \tilde{u}^{j+1} \right) - \partial_i \tilde{u}^{j+1}.\partial_{ti} \tilde{u}^{j+1} \right] \mathrm{d}t\,\mathrm{d}y \\ &= \int_{\Gamma_{c_1,t,x}} \partial_t \left(\frac{\varepsilon}{2} |\partial_t \tilde{u}^{j+1}|^2 + \frac{\alpha+1}{2\alpha} |\nabla \tilde{u}^{j+1}|^2 \right) - \frac{\alpha+1}{\alpha} \sum_{i=1}^n \partial_i \left(\partial_i \tilde{u}^{j+1}.\partial_t \tilde{u}^{j+1} \right) \mathrm{d}t\,\mathrm{d}y. \end{aligned}$$

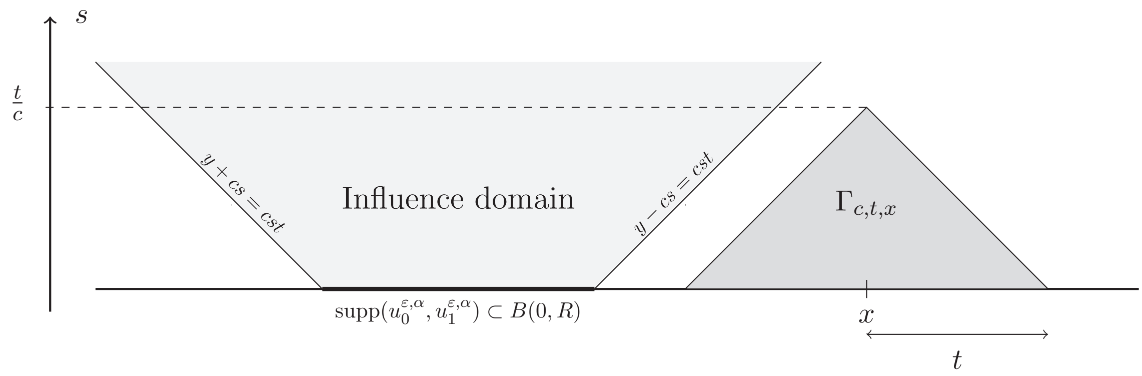


Figure 3.1 – The influence and dependence domains for an equation with speed of propagation c and with initial data supported in a ball $B(0,R)$.

In order to apply the divergence theorem, let us compute the unit outgoing normal ν to the cone $\Gamma_{c_i,t,x}$. We have

$$\nu(s,y) = \begin{cases} (+1,\mathbf{0}) & \text{if } s=t \\ (-1,\mathbf{0}) & \text{if } s=0 \\ \frac{1}{\sqrt{1+c_i^2}}(c_i, \frac{y-x}{|y-x|}) & \text{if } 0<s<t. \end{cases}$$

The divergence theorem yields :

$$\begin{aligned} E_{c_1}(\tilde{u}^{j+1})(0) &= E_{c_1}(\tilde{u}^{j+1})(t) + \frac{1}{\sqrt{1+c_1^2}} \int_0^{\frac{t}{c_1}} \int_{\mathbb{S}_{t-c_1 s}} c_1 e_{c_1}(\tilde{u}^{j+1})(s,y) - \\ &\quad - \frac{\alpha+1}{\alpha} \sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \left(\partial_i \tilde{u}^{j+1}.\partial_t \tilde{u}^{j+1}\right). \end{aligned}$$

By the Cauchy-Schwarz inequality followed by Young, we obtain :

$$\begin{aligned} \int \sum_{i=1}^n \frac{(y-x)_i}{|y-x|} \partial_i \tilde{u}^{j+1}.\partial_t \tilde{u}^{j+1} &\leq \sum_{i=1}^n \|\partial_i \tilde{u}^{j+1}\|_{L^2} \|\partial_t \tilde{u}^{j+1}\|_{L^2} \\ &\leq \frac{1}{2\lambda} \|\nabla \tilde{u}^{j+1}\|_{L^2}^2 + \frac{\lambda}{2} \|\partial_t \tilde{u}^{j+1}\|_{L^2}^2, \end{aligned}$$

where $\lambda = c_1^{-1}$. Since $\tilde{u}^{j+1}|_{t=0} = \operatorname{div} u_0$ and $\partial_t \tilde{u}^{j+1}|_{t=0} = \operatorname{div} u_1$, we know that

$$\tilde{u}^{j+1}, \partial_t \tilde{u}^{j+1} = 0 \text{ on } B(x,t).$$

So, finally, we have

$$E_{c_1}(\tilde{u}^{j+1})(t) \leq E_{c_1}(\tilde{u}^{j+1})(0) = 0$$

and we deduce that $\tilde{u}^{j+1},\ \partial_t\tilde{u}^{j+1} = 0$ on $\Gamma_{c_1,t,x}$.

Now, in order to handle the second equation in (S^j), we cover the cone $\Gamma_{c_1,x,t}$ with cones of the type Γ_{c_2,t_i,x_i} and integrate the second equation in (S^j) mutiplied by $\partial_t u^{j+1}$ on these cones. Doing so, we obtain that $u^{j+1} = 0$

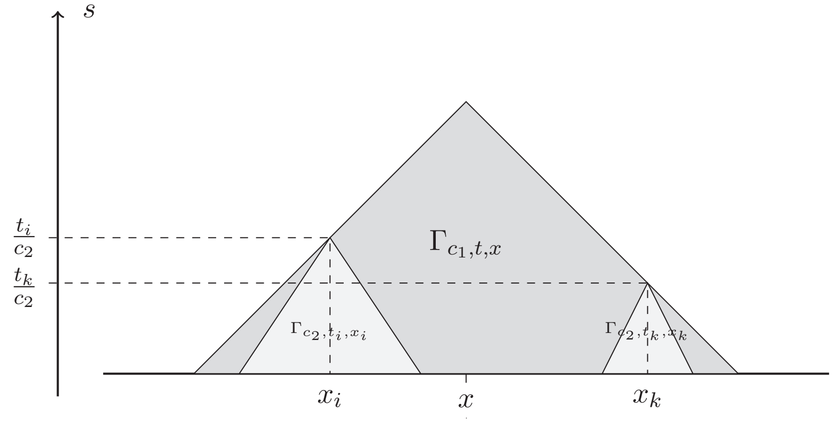


FIGURE 3.2 – Covering the cone $\Gamma_{c_1,x,t}$ with cones of the type Γ_{c_2,t_i,x_i}.

on $\Gamma_{c_1,t,x}$. We have proven that $(HNS^{\varepsilon,\alpha})$ has a finite speed of propagation $c(\varepsilon,\alpha) \geq c_1 \to +\infty$ as α goes to 0 .

3.4 Local existence for equation $(HNS^{\varepsilon,\alpha})$

3.4.1 Introduction

Let us consider the initial value problem

$$\left\{ \begin{array}{l} (HNS^{\varepsilon,\alpha})\ \ \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} - \frac{1}{\alpha}\nabla(\operatorname{div} u^{\varepsilon,\alpha}) = f(u^{\varepsilon,\alpha}) \\ u^{\varepsilon,\alpha}|_{t=0} = u_0^{\varepsilon,\alpha} \in H^{\frac{n}{2}+\delta}(\mathbb{R}^n)\ ,\ \ \partial_t u^{\varepsilon,\alpha}|_{t=0} = u_1^{\varepsilon,\alpha} \in H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \end{array} \right. , \tag{3.7}$$

where $n = 2, 3$ and $f(u) = -(u.\nabla)u$. First, let us assume that $f = 0$ and split the solution $u^{\varepsilon,\alpha}$ to equation $(HNS^{\varepsilon,\alpha})$ into its irrotational part $w^{\varepsilon,\alpha}$ and its divergence-free part $z^{\varepsilon,\alpha}$:

$$u^{\varepsilon,\alpha}(t,x) = w^{\varepsilon,\alpha}(t,x) + z^{\varepsilon,\alpha}(t,x).$$

More precisely, $w^{\varepsilon,\alpha}$ and $z^{\varepsilon,\alpha}$ are defined as follows :

$$\begin{aligned} w^{\varepsilon,\alpha} &= \mathbb{Q}u^{\varepsilon,\alpha} := \frac{1}{\Delta}\nabla(\operatorname{div} u^{\varepsilon,\alpha}) \\ z^{\varepsilon,\alpha} &= \mathbb{P}u^{\varepsilon,\alpha} := (\mathbf{1} - \mathbb{Q})u^{\varepsilon,\alpha} = u^{\varepsilon,\alpha} - w^{\varepsilon,\alpha}. \end{aligned}$$

Then $w^{\varepsilon,\alpha}$ and $z^{\varepsilon,\alpha}$ solve the following equations :

$$\begin{cases} \varepsilon\partial_{tt}w^{\varepsilon,\alpha} + \partial_t w^{\varepsilon,\alpha} - \left(1+\frac{1}{\alpha}\right)\Delta w^{\varepsilon,\alpha} = 0 \\ w^{\varepsilon,\alpha}|_{t=0} = w_0^{\varepsilon,\alpha} := \mathbb{Q}u_0^{\varepsilon,\alpha}\ ,\quad \partial_t w^{\varepsilon,\alpha}|_{t=0} = w_1^{\varepsilon,\alpha} := \mathbb{Q}u_1^{\varepsilon,\alpha}, \end{cases} \tag{3.8}$$

$$\begin{cases} \varepsilon\partial_{tt}z^{\varepsilon,\alpha} + \partial_t z^{\varepsilon,\alpha} - \Delta z^{\varepsilon,\alpha} = 0 \\ z^{\varepsilon,\alpha}|_{t=0} = z_0^{\varepsilon,\alpha} := \mathbb{P}u_0^{\varepsilon,\alpha}\ ,\quad \partial_t z^{\varepsilon,\alpha}|_{t=0} = z_1^{\varepsilon,\alpha} := \mathbb{P}u_1^{\varepsilon,\alpha}. \end{cases} \tag{3.9}$$

Since (3.8) is a wave equation, we know that

$$w^{\varepsilon,\alpha}(t,x) = A_{\mathbb{Q}}(t)w_0^{\varepsilon,\alpha}(x) + B_{\mathbb{Q}}(t)w_1^{\varepsilon,\alpha}(x) - \int_0^t B_{\mathbb{Q}}(t-s)\partial_t w^{\varepsilon,\alpha}(s)\,ds,$$

where $A_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ are defined as follows.

$$A_{\mathbb{Q}}(t) = \cos\left(\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}t\Lambda\right)\ ,\quad B_{\mathbb{Q}}(t) = \frac{\sin\left(\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}t\Lambda\right)}{\sqrt{\frac{\alpha+1}{\varepsilon\alpha}}\Lambda}.$$

Similarly, (3.9) is a wave equation and we have

$$z^{\varepsilon,\alpha}(t,x) = A_{\mathbb{P}}(t)z_0^{\varepsilon,\alpha}(x) + B_{\mathbb{P}}(t)z_1^{\varepsilon,\alpha}(x) - \int_0^t B_{\mathbb{P}}(t-s)\partial_t z^{\varepsilon,\alpha}(s)\,ds,$$

where $A_{\mathbb{P}}$ and $B_{\mathbb{P}}$ are

$$A_{\mathbb{P}}(t) = \cos\left(\frac{t\Lambda}{\sqrt{\varepsilon}}\right)\ ,\quad B_{\mathbb{P}}(t) = \sqrt{\varepsilon}\frac{\sin\left(\frac{t\Lambda}{\sqrt{\varepsilon}}\right)}{\Lambda}.$$

Now, setting $A(t) = A_{\mathbb{Q}}(t)\mathbb{Q} + A_{\mathbb{P}}(t)\mathbb{P}$ and $B(t) = B_{\mathbb{Q}}(t)\mathbb{Q} + B_{\mathbb{P}}(t)\mathbb{P}$, we can write Duhamel's formula for the initial value problem (3.7) with $f = 0$:

$$\phi(u^{\varepsilon,\alpha})(t) = A(t)u_0^{\varepsilon,\alpha} + B(t)u_1^{\varepsilon,\alpha} - \int_0^t B(t-s)\partial_t u^{\varepsilon,\alpha}(s)\,ds.$$

Adding the source term $f(u^{\varepsilon,\alpha}) = -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}$, the formula becomes

$$\phi(u^{\varepsilon,\alpha})(t) = A(t)u_0^{\varepsilon,\alpha} + B(t)u_1^{\varepsilon,\alpha} + \int_0^t B(t-s)\left(f(u^{\varepsilon,\alpha}) - \partial_t u^{\varepsilon,\alpha}\right)(s)\,ds.$$

3.4.2 Contraction argument

We shall show local existence for (3.7) in the complete metric space

$$\begin{aligned} X_T(a) &= \Big\{(u,\partial_t u) \in \dot H^{\frac{n}{2}+\delta} \cap \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \times \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n) \ : \\ &\qquad \|u\|_{X_T} := \|u\|_{L^\infty_T \dot H^{\frac{n}{2}+\delta}} + \|u\|_{L^\infty_T \dot H^{\frac{n}{2}+\delta-1}} + \|\partial_t u\|_{L^\infty_T \dot H^{\frac{n}{2}+\delta-1}} \le a\Big\}, \end{aligned}$$

with $a>0$ and $0<T<1$ to be chosen later.
Let us estimate $\|\phi(u)\|_{X_T}$. First, we have

$$\begin{aligned} \|\phi(u)(t)\|_{\dot H^{\frac{n}{2}+\delta}} &\le \|A_{\mathbb{Q}}\mathbb{Q}u_0\|_{\dot H^{\frac{n}{2}+\delta}} + \|A_{\mathbb{P}}\mathbb{P}u_0\|_{\dot H^{\frac{n}{2}+\delta}} + \|B_{\mathbb{Q}}\mathbb{Q}u_1\|_{\dot H^{\frac{n}{2}+\delta}} + \|B_{\mathbb{P}}\mathbb{P}u_1\|_{\dot H^{\frac{n}{2}+\delta}} \\ &\quad + \int_0^t \|B_{\mathbb{Q}}(t-s)\mathbb{Q}\left(f(u)-\partial_t u\right)\|_{\dot H^{\frac{n}{2}+\delta}}\,\mathrm{d}t \\ &\quad + \int_0^t \|B_{\mathbb{P}}(t-s)\mathbb{P}\left(f(u)-\partial_t u\right)\|_{\dot H^{\frac{n}{2}+\delta}}\,\mathrm{d}t \\ &\le 2\|u_0\|_{\dot H^{\frac{n}{2}+\delta}} + \left(1+\sqrt{\frac{\alpha}{\alpha+1}}\right)\sqrt{\varepsilon}\|u_1\|_{\dot H^{\frac{n}{2}+\delta-1}} \\ &\quad + \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}\int_0^t \left\| \frac{\sin\left(\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}(t-s)\Lambda\right)}{\Lambda}\ \mathbb{Q}\left[(u.\nabla)u-\partial_t u\right](s)\right\|_{\dot H^{\frac{n}{2}+\delta}} \mathrm{d}s \\ &\quad + \sqrt{\varepsilon}\int_0^t \left\| \frac{\sin\left(\frac{t-s}{\sqrt{\varepsilon}}\Lambda\right)}{\Lambda}\ \mathbb{P}\left[(u.\nabla)u-\partial_t u\right](s)\right\|_{\dot H^{\frac{n}{2}+\delta}} \mathrm{d}s + \\ &= 2\|u_0\|_{\dot H^{\frac{n}{2}+\delta}} + \left(1+\sqrt{\frac{\alpha}{\alpha+1}}\right)\sqrt{\varepsilon}\|u_1\|_{\dot H^{\frac{n}{2}+\delta-1}} + I + II. \end{aligned}$$

In order to estimate I, recall that $\operatorname{div} u \neq 0$, so that we have

$$(u.\nabla)u = \sum_{i=1}^n \partial_i\left(u_i u\right) - \operatorname{div} u\ u.$$

Using that $\dot H^{\frac{n}{2}+\delta} \cap \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)$ is an algebra, we obtain the inequality

$$\|\partial_i\left(u_i u\right)\|_{\dot H^{\frac{n}{2}+\delta-1}} \le \|u_i u\|_{\dot H^{\frac{n}{2}+\delta}} \le 2C\|u\|_{\dot H^{\frac{n}{2}+\delta}}\|u\|_{\dot H^{\frac{n}{2}+\delta-1}}.$$

Now, the following Lemma allows to estimate the remaining term.

Lemma 3.1. *Let $u \in \dot H^{\frac{n}{2}+\delta} \cap \dot H^{\frac{n}{2}+\delta-1}(\mathbb{R}^n)$ and $div u \neq 0$. Then we have*

$$\|div u\ u\|_{\dot H^{\frac{n}{2}+\delta-1}} \le 2C\|u\|_{\dot H^{\frac{n}{2}+\delta}}\|u\|_{L^\infty}.$$

Proof. Applying the paraproduct decomposition (see Lemma B.1 in the appendix), we can write

$$\begin{aligned}
\|\partial_i u\ u\|_{\dot H^{\frac{n}{2}+\delta-1}} &\leq \sum_p 2^{p\left(\frac{n}{2}+\delta-1\right)}\|\Delta_p\partial_i u\ S_{p+1}u\|_{L^2} \\
&\quad + \sum_q 2^{q\left(\frac{n}{2}+\delta-1\right)}\|\Delta_q u\ S_{p+1}\partial_i u\|_{L^2} \\
&\leq \sum_p 2^{p\left(\frac{n}{2}+\delta-1\right)}\|\Delta_p\partial_i u\|_{L^2}\|S_{p+1}u\|_{L^\infty} \\
&\quad + \sum_q 2^{q\left(\frac{n}{2}+\delta-1\right)}\|\Delta_q u\|_{L^2}\|S_{p+1}\partial_i u\|_{L^\infty} \\
&\leq \sum_p 2^{p\left(\frac{n}{2}+\delta\right)}\|\Delta_p u\|_{L^2}\|u\|_{L^\infty} + \sum_q 2^{q\left(\frac{n}{2}+\delta\right)}\|\Delta_q u\|_{L^2}\|u\|_{L^\infty} \\
&\leq 2C\|u\|_{\dot H^{\frac{n}{2}+\delta}}\|u\|_{L^\infty}.
\end{aligned}$$

The Lemma is thereby proved. □

Now, using the interpolation estimate $\|u\|_{L^\infty} \leq C\|u\|^{\delta}_{\dot H^{\frac{n}{2}+\delta-1}}\|u\|^{1-\delta}_{\dot H^{\frac{n}{2}+\delta}}$ followed by a Young inequality, we obtain that

$$\|\mathrm{div}\, u\ u\|_{\dot H^{\frac{n}{2}+\delta-1}} \leq 2C\|u\|_{\dot H^{\frac{n}{2}+\delta-1}}\|u\|_{\dot H^{\frac{n}{2}+\delta}} + \|u\|^2_{\dot H^{\frac{n}{2}+\delta}}.$$

Finally, we have

$$\|(u.\nabla)u\|_{\dot H^{\frac{n}{2}+\delta-1}} \leq 4C\|u\|^2_{\dot H^{\frac{n}{2}+\delta}\cap\dot H^{\frac{n}{2}+\delta-1}}.$$

and the integral I estimates

$$\begin{aligned}
I &\leq \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}\int_0^t \left(\|\mathbb{Q}(u.\nabla)u(s)\|_{\dot H^{\frac{n}{2}+\delta-1}} + \|\mathbb{Q}\partial_t u(s)\|_{\dot H^{\frac{n}{2}+\delta-1}}\right)\,\mathrm{d}s \\
&\leq \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}\int_0^t \left(\|(u.\nabla)u(s)\|_{\dot H^{\frac{n}{2}+\delta-1}} + \|\partial_t u(s)\|_{\dot H^{\frac{n}{2}+\delta-1}}\right)\,\mathrm{d}s \\
&\leq \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}\int_0^t \left(4C\|u(s)\|^2_{\dot H^{\frac{n}{2}+\delta}\cap\dot H^{\frac{n}{2}+\delta-1}} + \|\partial_t u(s)\|_{\dot H^{\frac{n}{2}+\delta-1}}\right)\,\mathrm{d}s \\
&\leq \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}\left(4CT\|u(s)\|^2_{L^\infty_T\left(\dot H^{\frac{n}{2}+\delta}\cap\dot H^{\frac{n}{2}+\delta-1}\right)} + T\|\partial_t u(s)\|_{L^\infty_T\dot H^{\frac{n}{2}+\delta-1}}\right) \\
&\leq \sqrt{\frac{\varepsilon\alpha}{\alpha+1}}T(4Ca^2+a).
\end{aligned}$$

Similarly, we obtain

$$II \leq \sqrt{\varepsilon}T(4Ca^2 + a).$$

So, finally, we have the inequality

$$\begin{aligned}\|\phi(u)\|_{L^\infty_T \dot{H}^{\frac{n}{2}+\delta}} &\leq 2\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta}} + \left(1 + \sqrt{\frac{\alpha}{\alpha+1}}\right)\sqrt{\varepsilon}\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \\ &+\sqrt{\varepsilon}\left(1 + \sqrt{\frac{\alpha}{\alpha+1}}\right) aT(1+4Ca).\end{aligned}$$

Analogous estimates using that $\left|\frac{\sin x}{x}\right| \leq 1$ give

$$\|\phi(u)\|_{L^\infty_T \dot{H}^{\frac{n}{2}+\delta-1}} \leq 2\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + 2T\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + 2aT^2(1+4Ca).$$

Then, let us compute the time derivative of $\phi(v)$.

$$\begin{aligned}\partial_t\phi(v)(t) &= -\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\Lambda\sin\left(\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}t\Lambda\right)\mathbb{Q}v_0 - \frac{\Lambda}{\sqrt{\varepsilon}}\sin\left(\frac{t}{\sqrt{\varepsilon}}\Lambda\right)\mathbb{P}v_0 + \\ &+\cos\left(\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}t\Lambda\right)\mathbb{Q}v_1 \qquad\qquad + \cos\left(\frac{t}{\sqrt{\varepsilon}}\Lambda\right)\mathbb{P}v_1 + \\ &+\int_0^t \cos\left(\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}(t-s)\Lambda\right)\mathbb{Q}\left(v.\nabla v - \partial_t v\right)(s)\,\mathrm{d}s + \\ &+\int_0^t \cos\left(\frac{t-s}{\sqrt{\varepsilon}}\Lambda\right)\mathbb{P}\left(v.\nabla v - \partial_t v\right)(s)\,\mathrm{d}s.\end{aligned}$$

We have

$$\|\partial_t\phi(u)\|_{L^\infty_T \dot{H}^{\frac{n}{2}+\delta-1}} \leq \frac{1}{\sqrt{\varepsilon}}\left(1 + \sqrt{\frac{\alpha+1}{\alpha}}\right)\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + 2aT(1+4Ca).$$

Finally, we obtain

$$\begin{aligned}\|\phi(u)\|_{X_T} &\leq \left(2 + \frac{1}{\sqrt{\varepsilon}} + \sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta}} + 2\|u_0\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \\ &+\left(2 + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + 2T\|u_1\|_{\dot{H}^{\frac{n}{2}+\delta-1}} + \\ &+\left(2 + 2T + \sqrt{\varepsilon} + \sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right) aT(1+4Ca).\end{aligned}$$

Let us set

$$\begin{aligned}\frac{a}{2} &= \left(2+\frac{1}{\sqrt{\varepsilon}}+\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0\|_{\dot H^{\frac{n}{2}+\delta}}+2\|u_0\|_{\dot H^{\frac{n}{2}+\delta-1}}\\ &\quad+\left(2+\sqrt{\varepsilon}+\sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1\|_{\dot H^{\frac{n}{2}+\delta-1}}.\end{aligned}$$

So we have

$$\|\phi(u)\|_{X_T}\leq\frac{a}{2}+\frac{aT}{2+\sqrt{\varepsilon}+\sqrt{\frac{\alpha\varepsilon}{\alpha+1}}}+\left(2+2T+\sqrt{\varepsilon}+\sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)aT(1+4Ca).$$

Finally, we have the following bound on the local existence time :

$$T\leq\frac{C}{1+\left[\left(C_\varepsilon+\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta}}+2\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta-1}}+\left(\tilde C_\varepsilon+\sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta-1}}\right]}.$$

In order to prove that the solutions obtained in this section are global, we shall prove that the denominator

$$\left(C_\varepsilon+\sqrt{\frac{\alpha+1}{\alpha\varepsilon}}\right)\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta}}+2\|u_0^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta-1}}+\left(\tilde C_\varepsilon+\sqrt{\frac{\alpha\varepsilon}{\alpha+1}}\right)\|u_1^{\varepsilon,\alpha}\|_{\dot H^{\frac{n}{2}+\delta-1}}$$

remains bounded for all fixed α and ε.

In subsections 3.5.2 and 3.6.2,we will globalize the solutions obtained in this section and then, in subsections 3.5.3 and 3.6.3, we will prove that they converge towards the solutions to (HNS^ε) as α goes to 0.

3.5 The 2D case

3.5.1 Preliminary estimates

In this part, we shall recall the regularity results we have on the solution u^ε to (HNS^ε).

Lemma 3.2. *Let $T>0$ and $u^\varepsilon\in L^\infty_T(\dot H^{1+\delta}\cap\dot H^\delta)(\mathbb{R}^2)^2$ be the global solution to (HNS^ε) with initial data $(u_0^\varepsilon,u_1^\varepsilon)\in H^{1+\delta}\times H^\delta(\mathbb{R}^2)^2$. Then we have*

$$u^\varepsilon\in L^\infty_T\dot H^{1+\delta}\cap L^2_T\dot H^{1+\delta}\cap L^2_TL^2\cap L^\infty_TL^2\cap L^2_T\dot H^1$$

and

$$\partial_t u^\varepsilon\in L^2_TL^2.$$

We sketch here the proof of this lemma since all the details can be found in [20]. In the following, we shall denote by C all the constants, even those depending on T.

Proof. First, let us introduce the energy

$$E_\varepsilon^\delta(t) := \frac{1}{2}\|u^\varepsilon + \varepsilon\partial_t u^\varepsilon\|_{\dot H^\delta}^2 + \frac{\varepsilon^2}{2}\|\partial_t u^\varepsilon\|_{\dot H^\delta}^2 + \varepsilon\|\nabla u^\varepsilon\|_{\dot H^\delta}^2,$$

for non-negative δ. Then, according to [20], we know that

$$\exists C > 0 \ : \ \forall t \geq 0\ , \quad E_\varepsilon^\delta(t) \leq C\varepsilon^{-\delta}.$$

From this inequality, we immediately deduce that

$$\varepsilon\|u^\varepsilon\|_{L^\infty_T \dot H^{1+\delta}} + \varepsilon\|u^\varepsilon\|_{L^2_T \dot H^{1+\delta}} \leq C\varepsilon^{-\delta} \text{ and } \|u^\varepsilon\|_{L^\infty_T L^\infty} = o\left(\frac{1}{\sqrt{\varepsilon}}\right). \qquad (3.10)$$

Now, let us compute the time derivative of E_ε^0. We have

$$\frac{d}{dt}E_\varepsilon^0(t)+\varepsilon\int_{\mathbb{R}^2}|\partial_t u^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2\,dx+\int_{\mathbb{R}^2}\left(|\nabla u^\varepsilon|^2-\varepsilon|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2\right)\,dx=0.$$

Due to the control of the norm $\|u^\varepsilon\|_{L^\infty_T L^\infty}$ by $\dfrac{1}{C\sqrt{\varepsilon}}$ for any C provided that ε is small enough, the last term in the left hand side is lower bounded by $\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x$. Now, integrating in time, we have

$$E_\varepsilon^0(T)+\int_0^T\int_{\mathbb{R}^2}\varepsilon|\partial_t u^\varepsilon+\nabla:(u^\varepsilon\otimes u^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \leq E_\varepsilon^0(0)\leq C.$$

So, we have that $\varepsilon\|\partial_t u^\varepsilon + \nabla:(u^\varepsilon\otimes u^\varepsilon)\|_{L^2_T L^2}^2 \leq C$ and

$$\frac{1}{2}\|u^\varepsilon\|_{L^2_T \dot H^1}^2 \leq C. \qquad (3.11)$$

The later yields

$$\varepsilon\|\nabla:(u^\varepsilon\otimes u^\varepsilon)\|_{L^2_T L^2}^2 \leq C\varepsilon\|u^\varepsilon\|_{L^\infty_T L^\infty}^2\|u^\varepsilon\|_{L^2_T \dot H^1}^2 \leq C.$$

Consequently, we have

$$\|\sqrt{\varepsilon}\partial_t u^\varepsilon\|_{L^2_T L^2}^2 \leq 2\varepsilon\|\partial_t u^\varepsilon + \nabla:(u^\varepsilon\otimes u^\varepsilon)\|_{L^2_T L^2}^2 + 2\varepsilon\|\nabla:(u^\varepsilon\otimes u^\varepsilon)\|_{L^2_T L^2}^2 \leq C.$$

We have thereby proven that

$$\sqrt{\varepsilon}\partial_t u^\varepsilon \in L^2_T L^2 \text{ uniformly in } \varepsilon.$$

Finally, notice that $\|u^\varepsilon\|_{L^2}^2 \leq 2E_\varepsilon^0(t) \leq 2C_0$, so that

$$u^\varepsilon \in L^2_T L^2. \qquad (3.12)$$

3.5.2 Globalization

As in [20], let us define the energy

$$E^{\delta}_{\varepsilon,\alpha}(t) = \int_{\mathbb{R}^2} \frac{1}{2}|\Lambda^{\delta}(u^{\varepsilon,\alpha}+\varepsilon\partial_t u^{\varepsilon,\alpha})|^2+\frac{\varepsilon^2}{2}|\Lambda^{\delta}\partial_t u^{\varepsilon,\alpha}|^2+\varepsilon|\Lambda^{\delta}\nabla u^{\varepsilon,\alpha}|^2+\frac{\varepsilon}{\alpha}|\Lambda^{\delta}\mathrm{div}\, u^{\varepsilon,\alpha}|^2.$$

In view of the dependence of the local time existence on the initial data, we know that proving that $E^{\delta}_{\varepsilon,\alpha}$ is bounded yields the global existence for $(HNS^{\varepsilon,\alpha})$. We shall first show that it is true on a time interval $[0,T)$ then prove that $T=+\infty$.

First, let us point out that $\dot{H}^{1+\delta}\cap L^{\infty}(\mathbb{R}^2)$ is an algebra and that the product estimate

$$\|fg\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} \leq C_1\left(\|f\|_{\dot{H}^{1+\delta}}\|g\|_{\infty}+\|g\|_{\dot{H}^{1+\delta}}\|f\|_{\infty}\right). \tag{3.13}$$

holds (see Proposition 1 in the appendix or [1]) for all functions $f,g\in\dot{H}^{1+\delta}\cap L^{\infty}(\mathbb{R}^2)$. Moreover, we know that the homogeneous Besov [1] space $\dot{B}^1_{2,1}(\mathbb{R}^2)$ embeds into $L^{\infty}(\mathbb{R}^2)$ and, interpolating, we obtain

$$\|f\|_{\infty} \;\leq\; \tilde{C}\|f\|_{\dot{B}^1_{2,1}} \;\leq\; C_2\,\|f\|^{\delta}_{\dot{H}^{\delta}}\,.\,\|f\|^{1-\delta}_{\dot{H}^{1+\delta}}\,. \tag{3.14}$$

Finally, notice that $\mathrm{div}\, u^{\varepsilon,\alpha}\neq 0$, so that we have

$$(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} = \sum_{i=1}^{2} u_i^{\varepsilon,\alpha}\partial_i u^{\varepsilon,\alpha} = \sum_{i=1}^{2}\partial_i(u_i^{\varepsilon,\alpha}.u^{\varepsilon,\alpha}) - \mathrm{div}\, u^{\varepsilon,\alpha}\times u^{\varepsilon,\alpha}. \tag{3.15}$$

but we still can prove the estimate

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot{H}^{\delta}} \leq C_3\|u^{\varepsilon,\alpha}\|_{L^{\infty}}\|u^{\varepsilon,\alpha}\|_{\dot{H}^{1+\delta}} \tag{3.16}$$

using that $\dot{H}^{1+\delta}\cap L^{\infty}(\mathbb{R}^2)$ is an algebra for the first term in (3.15) and due to Lemma 3.1 for the second term.

Now, let us define $T^{\max}$ the maximal existence time of $(HNS^{\varepsilon,\alpha})$ and prove the following lemma.

Lemma 3.3. *Assume the following, when ε goes to zero :*

$$(H)\left\{\begin{array}{ll} i) & \varepsilon^{\frac{1+\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{1+\delta}}+\varepsilon^{\frac{\delta}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\delta}} = o(1)\\ ii) & \varepsilon^{\frac{1}{2}}\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{1}}+\varepsilon\|u_1^{\varepsilon,\alpha}\|_{L^2} = o(1).\end{array}\right.$$

1. For definitions and properties of the Besov spaces, see the book by P.-G. Lemarié-Rieusset [28]

Let us define $0 \le T \le T^{\max}$ *by*

$$T = \sup \left\{ 0 \le \tau \le T^{\max} \ : \ \forall\, t \in [0,\tau), \ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty} < \frac{1}{2C_3\sqrt{\varepsilon}} \right\}. \tag{3.17}$$

Then, for ε *small enough, there exists a large number* N*, depending only on* δ *and* $\|u_0^{\varepsilon,\alpha}\|_{L^2}$ *(which is arbitrary), such that, for all* $0 \le t < T$*,*

$$E^\delta_{\varepsilon,\alpha}(t) \le E^\delta_{\varepsilon,\alpha}(0) \left(2\|u_0^{\varepsilon,\alpha}\|^2_{L^2} + 1\right)^N. \tag{3.18}$$

Proof. Let us compute the time derivative of $E^\delta_{\varepsilon,\alpha}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E^\delta_{\varepsilon,\alpha}(t) &= \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} - \|\nabla u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x - \\
&\quad - \varepsilon\|\partial_t u^{\varepsilon,\alpha} + (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} - \frac{1}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} \\
&\le \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} - \|\nabla u^{\varepsilon,\alpha}\|^2_{\dot H^\delta} - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x.
\end{aligned}$$

Using (3.16), the derivative of the energy estimates as follows :

$$\frac{\mathrm{d}}{\mathrm{d}t} E^\delta_{\varepsilon,\alpha}(t) \le \left(C_3^2 \varepsilon \|u^{\varepsilon,\alpha}\|^2_\infty - 1\right) \|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}} - \int_{\mathbb{R}^2} \Lambda^\delta u^{\varepsilon,\alpha}.\Lambda^\delta (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x.$$

Since we assume $i)$ and using inequality (3.14), we can write, for ε small enough,

$$\|u_0^{\varepsilon,\alpha}\|_{L^\infty} \le \frac{1}{2C_3\sqrt{\varepsilon}}. \tag{3.19}$$

Now, by continuity of the (local) solution $u^{\varepsilon,\alpha}$ with respect to t, we deduce that $T > 0$ and that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E^\delta_{\varepsilon,\alpha}(t) \le -\frac{1}{4}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}} + C\|u^{\varepsilon,\alpha}\|_{\dot H^\delta}\|u^{\varepsilon,\alpha}\|_\infty \|u^{\varepsilon,\alpha}\|_{\dot H^{1+\delta}}$$

holds on $[0,T)$. Then, using the interpolation inequalities

$$\|u^{\varepsilon,\alpha}\|_{\dot H^\delta} \le C_4 \|u^{\varepsilon,\alpha}\|^{1-\delta}_{L^2} \|u^{\varepsilon,\alpha}\|^\delta_{\dot H^1} \tag{3.20}$$

and (3.14), we obtain the estimate

$$\frac{d}{dt} E^\delta_{\varepsilon,\alpha}(t) \le -\frac{1}{4}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{1+\delta}} + C\|u^{\varepsilon,\alpha}\|^{1-\delta}_2 \left(\|u^{\varepsilon,\alpha}\|_{\dot H^1}\|u^{\varepsilon,\alpha}\|_{\dot H^\delta}\right)^\delta \|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot H^{1+\delta}}.$$

Finally, a Young inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E^\delta_{\varepsilon,\alpha}(t) \le C\|u^{\varepsilon,\alpha}\|_2^{2\frac{1-\delta}{\delta}} \|u^{\varepsilon,\alpha}\|^2_{\dot H^1} E^\delta_{\varepsilon,\alpha}(t). \tag{3.21}$$

In order to show that $E^{\delta}_{\varepsilon,\alpha}$ is bounded, we shall use the decay of $E^{0}_{\varepsilon,\alpha}$. So, let us estimate its time derivative :

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(t) \leq \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{L^2}-\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1}-\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x-\frac{1}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{L^2}.$$

First, by Lemma 3.1, we immediately have

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2} \leq C_3\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1}.$$

Besides, by integrations by parts, we obtain

$$\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.\,\mathrm{div}\, u^{\varepsilon,\alpha}\ u^{\varepsilon,\alpha} = -\frac{1}{2}\int_{\mathbb{R}^2}\mathrm{div}\, u^{\varepsilon,\alpha}\ |u^{\varepsilon,\alpha}|^2.$$

Recall that $\mathrm{div}\, u^{\varepsilon,\alpha} \in L^2(\mathbb{R}^2)$ and $u^{\varepsilon,\alpha} \in L^2 \cap \dot{H}^1 \subset \dot{H}^{\frac{1}{2}} \subset L^4(\mathbb{R}^2)$. Thus the integral estimates

$$\begin{aligned}
|\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x| &= |\frac{1}{2}\int_{\mathbb{R}^2}\mathrm{div}\, u^{\varepsilon,\alpha}\ |u^{\varepsilon,\alpha}|^2| \\
&\leq \frac{1}{2}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\||u^{\varepsilon,\alpha}|^2\|_{L^2} \\
&\leq \frac{1}{2}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|^2_{L^4} \\
&\leq \frac{K}{2}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha}\|_{\dot{H}^1} \\
&\leq \frac{K^2}{8}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{L^2}\|u^{\varepsilon,\alpha}\|^2_{L^2}+\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1},
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality

$$\|f\|^2_{L^4} \leq K\|f\|_{L^2}\|f\|_{\dot{H}^1}$$

for $f \in H^1(\mathbb{R}^2)$. We therefore obtain the following estimate on $E^0_{\varepsilon,\alpha}$ on $[0,T)$ for ε small enough :

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(t) &\leq \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{L^2}-\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1} \\
&\quad -\int_{\mathbb{R}^2} u^{\varepsilon,\alpha}.(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\,\mathrm{d}x-\frac{1}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{L^2} \\
&\leq \left(C_3^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty}-\frac{1}{2}\right)\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1} \\
&\quad +\left(\frac{K^2}{8}\|u^{\varepsilon,\alpha}\|^2_{L^2}-\frac{1}{\alpha}\right)\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{L^2} \\
&\leq -\frac{1}{4}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1}+\left(\frac{K^2}{8}\|u^{\varepsilon,\alpha}\|^2_{L^2}-\frac{1}{\alpha}\right)\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{L^2}.
\end{aligned}$$

Moreover, assuming that $\alpha \leq \dfrac{2}{K^2\|u_0^{\varepsilon,\alpha}\|_{L^2}^2}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\varepsilon,\alpha}(0) \leq -\frac{1}{4}\|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^1}.$$

Besides, the smallness assumptions *ii)* on the initial data yield

$$\|u^{\varepsilon,\alpha}(t)\|^2_{L^2} \ \leq\ 2E^0_{\varepsilon,\alpha}(t) \ \leq\ 2E^0_{\varepsilon,\alpha}(0) \ \overset{ii)}{\leq}\ 4\|u_0^{\varepsilon,\alpha}\|^2_{L^2}$$

and $E^0_{\varepsilon,\alpha}$ is therefore decreasing on $[0,T)$.
Let us now define the functional $\mathcal{E}^\delta_{\varepsilon,\alpha} := E^\delta_{\varepsilon,\alpha}(1+E^0_{\varepsilon,\alpha})^N$. One can easily check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\delta_{\varepsilon,\alpha} \leq E^\delta_{\varepsilon,\alpha}(1+E^0_{\varepsilon,\alpha})^{N-1}\|u^{\varepsilon,\alpha}\|^2_{\dot{H}^1}\left(C\|u_0^{\varepsilon,\alpha}\|_{L^2}^{2\frac{1-\delta}{\delta}}(1+2\|u_0^{\varepsilon,\alpha}\|^2_{L^2}) - \frac{N}{4}\right).$$

So, if the positive integer $N = N(\delta, \|u_0^{\varepsilon,\alpha}\|_2)$ is large enough, $\mathcal{E}^\delta_{\varepsilon,\alpha}$ decreases and $E^\delta_{\varepsilon,\alpha}(t)$ satisfies the estimate (3.18) for all $t \in [0,T)$. □

The aim of the following lemma is to ensure the control of $\|u^{\varepsilon,\alpha}(t)\|_\infty$ throughout the time, so that we can reiterate the reasoning.

Lemma 3.4. *Assume that*

$$\varepsilon^{1+\frac{\delta}{2}}\|u_1^{\varepsilon,\alpha}\|_{\dot{H}^\delta} \longrightarrow 0\ , \quad \varepsilon \to 0 \tag{3.22}$$

in addition to the assumptions (H) *in Lemma 3.3. Then the inequality*

$$E^\delta_{\varepsilon,\alpha}(0) \ <\ C_\delta(2\|u_0^{\varepsilon,\alpha}\|_2^2+1)^{-N}. \tag{3.23}$$

holds for ε *small enough. Moreover, for all* $t \in [0,T)$*,*

$$\|u^{\varepsilon,\alpha}\|_{L^\infty} \ \leq\ \frac{1}{4C_3\sqrt{\varepsilon}}. \tag{3.24}$$

We skip the proof of this lemma since it can be found in [20].

Remark. *Notice that, under the assumptions* (3.1) *in Theorem 3.2, the conditions* (H) *in Lemma 3.3 and* (3.22) *in Lemma 3.4 are fulfilled.*

As a consequence of Lemmas 3.3 and 3.4, we obtain that E^δ_ε satisfies inequality (2.14) on the whole existence interval $[0,T^{\max})$. Therefore $(HNS^{\varepsilon,\alpha})$ has a global solution.

3.5.3 Convergence

Let $u^{\varepsilon,\alpha}$ and u^ε be the solutions to

$$\begin{aligned}
(HNS^{\varepsilon,\alpha}) \quad \varepsilon\partial_{tt}u^{\varepsilon,\alpha} + \partial_t u^{\varepsilon,\alpha} - \Delta u^{\varepsilon,\alpha} &= -(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla \mathrm{div}\, u^{\varepsilon,\alpha}\\
(HNS^{\varepsilon}) \quad \varepsilon\partial_{tt}u^{\varepsilon} + \partial_t u^{\varepsilon} - \Delta u^{\varepsilon} &= -(u^{\varepsilon}.\nabla)u^{\varepsilon} + \nabla p^\varepsilon\ , \quad \mathrm{div}\, u^\varepsilon = 0
\end{aligned}$$

with the same initial data $u_0^{\varepsilon,\alpha} = u_0^\varepsilon \in H^{1+\delta}(\mathbb{R}^2)^2$ and $u_1^{\varepsilon,\alpha} = u_1^\varepsilon \in H^{\delta}(\mathbb{R}^2)^2$. Let us now define the following modulated energy

$$\begin{aligned}
E_{\varepsilon,\alpha,u^\varepsilon} &= \frac{1}{2}\|u^{\varepsilon,\alpha} - u^\varepsilon + \varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|_{L^2}^2 + \frac{\varepsilon^2}{2}\|\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|_{L^2}^2\\
&\quad +\varepsilon\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot H^1}^2 + \frac{\varepsilon}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}^2
\end{aligned}$$

which is inspired by the Dafermos modulated energy (see [7,20]). Notice that

$$E_{\varepsilon,\alpha,u^\varepsilon}\left(u^{\varepsilon,\alpha}\right) = E_{\varepsilon,\alpha}\left(u^{\varepsilon,\alpha} - u^\varepsilon\right).$$

Through a Gronwall estimate on this energy, we shall prove that $u^{\varepsilon,\alpha}$ converges to u^ε in the $L^\infty_T L^2(\mathbb{R}^2)$ norm, as α goes to 0. To this end, let us estimate the time derivative of the modulated energy $E_{\varepsilon,\alpha,u^\varepsilon}$ using the equations (HNS^ε) and $(HNS^{\varepsilon,\alpha})$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_{\varepsilon,\alpha,u^\varepsilon} &= \int_{\mathbb{R}^2}\Big(\left(\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon) + \varepsilon\partial_{tt}(u^{\varepsilon,\alpha} - u^\varepsilon)\right).\left(u^{\varepsilon,\alpha} - u^\varepsilon + \varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\right)\\
&\quad +\varepsilon^2\partial_{tt}(u^{\varepsilon,\alpha} - u^\varepsilon).\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon) + 2\varepsilon\partial_t\nabla(u^{\varepsilon,\alpha} - u^\varepsilon).\nabla(u^{\varepsilon,\alpha} - u^\varepsilon)\\
&\quad -\frac{2\varepsilon}{\alpha}\partial_t u^{\varepsilon,\alpha}.\nabla(\mathrm{div}\, u^{\varepsilon,\alpha})\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2}\Big(\left(\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon) + \varepsilon\partial_{tt}(u^{\varepsilon,\alpha} - u^\varepsilon)\right).\left(u^{\varepsilon,\alpha} - u^\varepsilon + 2\varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\right)\\
&\quad -\varepsilon|\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)|^2 - 2\varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon).\Delta(u^{\varepsilon,\alpha} - u^\varepsilon) -\\
&\quad -\frac{2\varepsilon}{\alpha}\partial_t u^{\varepsilon,\alpha}.\nabla(\mathrm{div}\, u^{\varepsilon,\alpha})\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^2}\Big(-|\nabla(u^{\varepsilon,\alpha} - u^\varepsilon)|^2 + \big((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big).(u^{\varepsilon,\alpha} - u^\varepsilon)\\
&\quad -\varepsilon|\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)|^2 + 2\varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon).\big((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big)\\
&\quad +\frac{1}{\alpha}\nabla(\mathrm{div}\, u^{\varepsilon,\alpha}).(u^{\varepsilon,\alpha} - u^\varepsilon) + \frac{2\varepsilon}{\alpha}\nabla(\mathrm{div}\, u^{\varepsilon,\alpha}).\left(\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon) - \partial_t u^{\varepsilon,\alpha}\right)\\
&\quad -\nabla p^\varepsilon.(u^{\varepsilon,\alpha} - u^\varepsilon) - 2\varepsilon\nabla p^\varepsilon.\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\Big)\,\mathrm{d}x.
\end{aligned}$$

Now, using that $\mathrm{div}\, u^\varepsilon = 0$ and the inequality

$$2\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon).\big((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\big) \leq |\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)|^2 + |(u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}|^2$$

and integrating in time, we finally obtain the estimate

$$\begin{aligned}
E_{\varepsilon,\alpha,u^\varepsilon}(T) \ \leq\ & E_{\varepsilon,\alpha,u^\varepsilon}(0) + \int_0^T\!\!\int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).\left((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\right) \mathrm{d}x\,\mathrm{d}t \\
& + \varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{L^2_T L^2} - \|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^2_T \dot{H}^1} \\
& - \int_0^T\!\!\int_{\mathbb{R}^2} \nabla p^\varepsilon.(u^{\varepsilon,\alpha} + 2\varepsilon\partial_t u^{\varepsilon,\alpha})\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

In the following subsections, we shall estimate the terms in the right hand side.

Notation : We shall write $f = \mathcal{O}(1)$ if there exists a constant $C = C(\varepsilon, u^\varepsilon)$ such that $f \leq C$. Similarly, $f = \mathcal{O}(\sqrt{\alpha})$ means that $f \leq C(\varepsilon, u^\varepsilon)\sqrt{\alpha}$.

3.5.3.1 Estimate on $\int_0^T \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).\left((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\right)\,\mathbf{d}x\mathbf{d}t$

From the estimates on the energies $E^0_{\varepsilon,\alpha}$ and E^0_ε, we know that

$$u^{\varepsilon,\alpha}, u^\varepsilon \in L^\infty_T L^2(\mathbb{R}^2) \cap L^2_T \dot{H}^1(\mathbb{R}^2)$$

and the boundedness of $E^0_{\varepsilon,\alpha}$ yields

$$\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^2} = \mathcal{O}(\sqrt{\alpha}).$$

Besides, let us recall Ladyzhenskaya's inequality :

$$\|f\|^2_{L^4} \leq c\|f\|_{L^2}\|f\|_{\dot{H}^1} \tag{3.25}$$

which holds for all $f \in H^1(\mathbb{R}^2)$. Hence we can estimate the integral

$$I = \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).\left((u^\varepsilon.\nabla)u^\varepsilon - (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\right)\,\mathrm{d}x$$

as follows :

$$\begin{aligned}
I \ =\ & \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).\big((u^\varepsilon - u^{\varepsilon,\alpha}).\nabla\big)u^\varepsilon\,\mathrm{d}x \\
& + \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).(u^{\varepsilon,\alpha}.\nabla)(u^\varepsilon - u^{\varepsilon,\alpha})\,\mathrm{d}x \\
=\ & \int_{\mathbb{R}^2} (u^{\varepsilon,\alpha} - u^\varepsilon).\big((u^\varepsilon - u^{\varepsilon,\alpha}).\nabla\big)u^\varepsilon\,\mathrm{d}x \\
& + \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{div}\, u^{\varepsilon,\alpha}\ |u^{\varepsilon,\alpha} - u^\varepsilon|^2\,\mathrm{d}x \\
\leq\ & \|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^4}\|\nabla u^\varepsilon\|_{L^2} + \frac{1}{2}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^4} \\
\leq\ & C\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{L^2}\|u^{\varepsilon,\alpha} - u^\varepsilon\|_{\dot{H}^1}\left(\|u^\varepsilon\|_{\dot{H}^1} + \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\right)
\end{aligned}$$

by (3.25). Now, using Young inequalities, we obtain that

$$\begin{aligned} I &\leq \frac{C^2}{\eta}\left(\|u^\varepsilon\|_{\dot H^1}+\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}\right)^2\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^2}^2+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^1}^2 \\ &\leq \frac{2C^2}{\eta}\left(\|u^\varepsilon\|_{\dot H^1}^2+\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2}^2\right)\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^2}^2+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^1}^2, \end{aligned}$$

where $\eta > 0$ is a small number to be chosen in the conclusion.

3.5.3.2 Estimate on $\varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L_T^2L^2}^2-\frac{1}{2}\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L_T^2\dot H^1}^2$

Let us set

$$A^{\varepsilon,\alpha}(t)=\varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2}^2-\frac{1}{2}\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^1}^2$$

and write

$$(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^\varepsilon.\nabla)u^\varepsilon=\big(u^{\varepsilon,\alpha}.\nabla\big)(u^{\varepsilon,\alpha}-u^\varepsilon)+\big((u^{\varepsilon,\alpha}-u^\varepsilon).\nabla\big)u^\varepsilon$$

then, using Lemma 3.1, estimate

$$\|(u^\varepsilon.\nabla)u^\varepsilon-(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{L^2} \leq \|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^1}+\|\big((u^{\varepsilon,\alpha}-u^\varepsilon).\nabla\big)u^\varepsilon\|_{L^2}.$$

The second term on the right hand side estimates using the Sobolev embeddings

$$\dot H^s\subset L^{\frac{2}{1-s}}(\mathbb{R}^2)$$

as follows. Notice that $\nabla u^\varepsilon\in L_T^\infty\dot H^\delta\subset L_T^\infty L^{\frac{2}{1-\delta}}$ and

$$u^{\varepsilon,\alpha}-u^\varepsilon\in L_T^2L^2\cap L_T^2\dot H^1\subset L_T^2\dot H^{1-\delta}\subset L_T^2L^{\frac{2}{\delta}}.$$

So we have

$$\begin{aligned} \|\big((u^{\varepsilon,\alpha}-u^\varepsilon).\nabla\big)u^\varepsilon\|_{L^2} &\leq \|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^{\frac{2}{\delta}}}\|\nabla u^\varepsilon\|_{L^{\frac{2}{1-\delta}}} \\ &\leq C\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^2}^\delta\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^1}^{1-\delta}\|\nabla u^\varepsilon\|_{\dot H^\delta}. \end{aligned}$$

Now, integrating in time the squared norm and performing a Young inequality, we obtain

$$\begin{aligned} \|\big(u^{\varepsilon,\alpha}-u^\varepsilon).\nabla\big)u^\varepsilon\|_{L_T^2L^2}^2 &\leq \frac{C_\delta}{\eta}\int_0^T\|u^\varepsilon\|_{\dot H^{1+\delta}}^2E_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t \\ &\quad+\tilde C_\delta\eta\|u^\varepsilon\|_{L_T^\infty\dot H^{1+\delta}}^2\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L_T^2\dot H^1}^2. \end{aligned}$$

Finally, recalling that $\varepsilon\left(\|u^\varepsilon\|^2_{L^\infty_T\dot H^{1+\delta}}+\|u^\varepsilon\|^2_{L^2_T\dot H^{1+\delta}}\right)=\mathcal{O}(1)$, we obtain the inequality

$$\begin{aligned}\int_0^T A^{\varepsilon,\alpha}\mathrm{d}t &\le \left(C\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty_T L^\infty}+\tilde C_\delta\varepsilon\eta\|u^\varepsilon\|^2_{L^\infty_T\dot H^{1+\delta}}-\frac12\right)\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^1}+\\ &\quad+\frac{C_\delta}{\eta}\int_0^T\varepsilon\|u^\varepsilon\|^2_{\dot H^{1+\delta}}E_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t\\ &\le \frac{C_\delta}{\eta}\int_0^T\varepsilon\|u^\varepsilon\|^2_{\dot H^{1+\delta}}E_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t\end{aligned}$$

if ε and η are small enough.

3.5.3.3 Estimate on $\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.u^{\varepsilon,\alpha}\,\mathbf{d}x\,\mathbf{d}t$

First, notice that, applying the div operator to equation (HNS^ε), we obtain the identity

$$\Delta p^\varepsilon=\mathrm{div}\,(u^\varepsilon.\nabla)u^\varepsilon=\mathrm{div}\,\nabla:u^\varepsilon\otimes u^\varepsilon \tag{3.26}$$

from which we deduce that p^ε has the same regularity as $u^\varepsilon\otimes u^\varepsilon$ and

$$\|p^\varepsilon\|_{L^2_TL^2}\le C\|u^\varepsilon\otimes u^\varepsilon\|_{L^2_TL^2}.$$

Since $u^\varepsilon\in L^2_T\dot H^1\cap L^\infty_TL^2\subset L^4_TL^4(\mathbb{R}^2)^2$, we immediately conclude that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T\int_{\mathbb{R}^2}p^\varepsilon.\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \|p^\varepsilon\|_{L^2_TL^2}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^2_TL^2}\\ &\le C\sqrt{\alpha}\|p^\varepsilon\|_{L^2_TL^2}\ \le\ C\sqrt{\alpha}\|u^\varepsilon\otimes u^\varepsilon\|_{L^2_TL^2}\\ &\le C\sqrt{\alpha}\|u^\varepsilon\|^2_{L^4_TL^4}\ =\ \mathcal{O}(\sqrt{\alpha}).\end{aligned}$$

3.5.3.4 Estimate on $2\varepsilon\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.\partial_t u^{\varepsilon,\alpha}\,\mathbf{d}x\,\mathbf{d}t$

From identity (3.26), we know that

$$\partial_t p^\varepsilon=\sum_{i,j=1}^2\partial_t\frac{\partial_i\partial_j}{\Delta}(u_i^\varepsilon u_j^\varepsilon).$$

Besides, due to the control of the energies E^0_ε and E^δ_ε, we have

$$\begin{aligned}\partial_t u^\varepsilon &\in L^2_TL^2\\ u^\varepsilon &\in L^\infty_TH^{1+\delta}\subset L^\infty_TL^\infty.\end{aligned}$$

After two integrations by parts (one in each variable), we obtain an integral which easily estimates :

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}\nabla p^\varepsilon.\partial_t u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^2}\partial_t p^\varepsilon.\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \|\partial_t p^\varepsilon\|_{L^2_TL^2}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^2_TL^2}\\ &\leq C\sqrt{\alpha}\|\partial_t u^\varepsilon\|_{L^2_TL^2}\|u^\varepsilon\|_{L^\infty_TL^\infty} = \mathcal{O}(\sqrt{\alpha}).\end{aligned}$$

3.5.3.5 Conclusion

First, notice that since we take the same initial data for $(HNS^{\varepsilon,\alpha})$ and (HNS^ε), we have in particular $\mathrm{div}\,u_0^{\varepsilon,\alpha}=0$ so

$$E_{\varepsilon,\alpha,u^\varepsilon}(0)=0.$$

Now, gathering the estimates in the previous subsections, we obtain that

$$\begin{aligned}E_{\varepsilon,\alpha,u^\varepsilon}(T) &\leq \mathcal{O}(\sqrt{\alpha})+C_{\delta,\eta}\int_0^T\left(\|u^\varepsilon\|^2_{\dot H^1}+\varepsilon\|u^\varepsilon\|^2_{\dot H^{1+\delta}}+\mathcal{O}(\alpha)\right)E_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t\\ &\quad+\left(\eta+(1-\delta)\varepsilon\eta\|u^\varepsilon\|^2_{L^\infty_T\dot H^{1+\delta}}-\frac{1}{2}\right)\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^1}.\end{aligned}$$

So, choosing η small enough, Gronwall's lemma yields, for all positive T,

$$E_{\varepsilon,\alpha,u^\varepsilon}(T)=\mathcal{O}(\sqrt{\alpha}).$$

Now, recall that Theorem 3.1 tells that u^ε converges towards the solution v to (NS). Theorem 3.2 is now proved.

3.6 The 3D case

3.6.1 Preliminary estimates

As in subsection 3.5.1, we start by recalling the regularity results we have on u^ε. All the proofs are in [20].

Lemma 3.5. *Let $u^\varepsilon\in L^\infty_T(\dot H^{\frac32+\delta}\cap\dot H^{\frac12+\delta})(\mathbb{R}^3)^3$ be the global solution to (HNS^ε) with initial data $(u_0^\varepsilon,u_1^\varepsilon)\in H^{\frac32+\delta}\times H^{\frac12+\delta}(\mathbb{R}^3)^3$. Then we have*

$$u^\varepsilon\in L^\infty_T\dot H^{\frac32+\delta}\cap L^2_T\dot H^{\frac32+\delta}\cap L^2_T\dot H^{\frac32}\cap L^\infty_T\dot H^{\frac12}\cap L^\infty_T\dot H^{\frac32}$$

and

$$\partial_t u^\varepsilon\in L^2_T\dot H^{\frac12}\cap L^\infty_T\dot H^{\frac12+\delta}.$$

Remark. *In this paper, we are not interested in the dependence of the norms on ε.*

Proof. Here is a sketch of the proof. First, let us define the 3D energy

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) = \frac{1}{2}\|u^\varepsilon + \varepsilon\partial_t u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}+\delta}} + \frac{\varepsilon^2}{2}\|\partial_t u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}+\delta}} + \varepsilon\|\nabla u^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}+\delta}}$$

for non-negative δ. In order to obtain that the solutions u^ε to (HNS^ε) are global, we proved in [20] that

$$\exists C > 0 \;:\; \forall t \geq 0\;, \quad E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq C\varepsilon^{-\delta}. \tag{3.27}$$

Directly from the expression of E_ε^δ and from (3.27), we deduce that

$$\partial_t u^\varepsilon \in L^\infty_T \dot{H}^{\frac{1}{2}+\delta}\;, \quad u^\varepsilon \in L^\infty_T \dot{H}^{\frac{3}{2}+\delta} \cap L^2_T \dot{H}^{\frac{3}{2}+\delta}.$$

Besides, let us consider the time derivative of $E_\varepsilon^{\frac{1}{2}}$. According to [20], we have

$$E_\varepsilon^{\frac{1}{2}}(T) + \varepsilon\|\partial_t u^\varepsilon + (u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{1}{2}}} + \|u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{3}{2}}} - \varepsilon\|(u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{1}{2}}} = E_\varepsilon^{\frac{1}{2}}(0) + I(\varepsilon,T),$$

where

$$\begin{aligned} I(\varepsilon,T) \;&:=\; \int_0^T\!\!\int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}\left((u^\varepsilon.\nabla)u^\varepsilon\right).\Lambda^{\frac{1}{2}}u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &\leq\; \|u^\varepsilon\|_{L^\infty_T\dot{H}^{\frac{1}{2}}}\|u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{3}{2}}}. \end{aligned}$$

The Λ above is the Fourier multiplier defined by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$.
So, since $E_\varepsilon^{\frac{1}{2}}(0) \leq C$ and $\|u^\varepsilon\|_{L^\infty_T\dot{H}^{\frac{1}{2}}} \leq 2\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{8}$ by the decay of $E_\varepsilon^{\frac{1}{2}}$, we obtain the inequality

$$E_\varepsilon^{\frac{1}{2}}(T) + \varepsilon\|\partial_t u^\varepsilon + (u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{1}{2}}} + \frac{1}{2}\|u^\varepsilon\|^2_{L^2_T\dot{H}^{\frac{3}{2}}} \leq C$$

if ε is small enough. Now, we complete the reasoning as in the 2D case and deduce that

$$\partial_t u^\varepsilon \in L^2_T\dot{H}^{\frac{1}{2}}\;, \quad u^\varepsilon \in L^2_T\dot{H}^{\frac{3}{2}}.$$

Finally, since $E_\varepsilon^{\frac{1}{2}}$ is bounded, we immediately have

$$u^\varepsilon \in L^\infty_T\dot{H}^{\frac{1}{2}} \cap L^\infty_T\dot{H}^{\frac{3}{2}}$$

3.6.2 Globalization

As in the 2D case, we shall prove that the energy

$$E^{\frac12+\delta}_{\varepsilon,\alpha}(t)=\frac12\|u^{\varepsilon,\alpha}+\varepsilon\partial_t u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}+\frac{\varepsilon^2}{2}\|\partial_t u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}+\varepsilon\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32+\delta}}+\frac{\varepsilon}{\alpha}\|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12+\delta}}$$

is bounded in order to show that the local solution obtained by the contraction argument is global. First, let us define $0\le T\le T^{\max}$ by

$$T=\sup\left\{0\le\tau\le T^{\max}\ :\ \forall\, t\in[0,\tau),\ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty}<\frac{1}{2K_1\sqrt{\varepsilon}}\right\},\tag{3.28}$$

where $K_1=\max(K,\tilde K)$ such that

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\ \le\ K\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32}},\tag{3.29}$$

$$\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|_{\dot H^{\frac12+\delta}}\ \le\ \tilde K\|u^{\varepsilon,\alpha}\|_{L^\infty}\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32+\delta}}.\tag{3.30}$$

We shall start by proving that the energy $E^{\frac12}_{\varepsilon,\alpha}$ decreases on $[0,T)$. To this end, we estimate its time derivative :

$$\begin{aligned}\frac{\mathrm{d}E^{\frac12}_{\varepsilon,\alpha}}{\mathrm{d}t}&\le\ \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac12}}-\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}-\int_{\mathbb R^3}\Lambda^{\frac12}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac12}u^{\varepsilon,\alpha}\,\mathrm{d}x\\&\le\ \left(K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty}-1\right)\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}-\int_{\mathbb R^3}\Lambda^{\frac12}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac12}u^{\varepsilon,\alpha}\,\mathrm{d}x.\end{aligned}$$

So, for ε small enough, we have

$$\frac{\mathrm{d}E^{\frac12}_{\varepsilon,\alpha}}{\mathrm{d}t}\le-\frac34\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}}+\int_{\mathbb R^3}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda u^{\varepsilon,\alpha}\,\mathrm{d}x$$

on $[0,T)$. Then, recalling that $\mathrm{div}\, u^{\varepsilon,\alpha}\neq0$, we have

$$\begin{aligned}\int_{\mathbb R^3}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda u^{\varepsilon,\alpha}\,\mathrm{d}x\ &=\ \sum_{i,j=1}^3\int_{\mathbb R^3}u_i^{\varepsilon,\alpha}\,\partial_i u_j^{\varepsilon,\alpha}\,\partial_j u_i^{\varepsilon,\alpha}\,\mathrm{d}x\\&=\ -\int_{\mathbb R^3}\left(\sum_{i,j=1}^3\partial_j u_i^{\varepsilon,\alpha}\,\partial_i u_j^{\varepsilon,\alpha}\,u_i^{\varepsilon,\alpha}+\sum_{i=1}^3(u_i^{\varepsilon,\alpha})^2\,\partial_i\mathrm{div}\, u^{\varepsilon,\alpha}\right)\mathrm{d}x\\&=\ \sum_{i=1}^3\int_{\mathbb R^3}u_i^{\varepsilon,\alpha}\,\partial_i u_i^{\varepsilon,\alpha}\ \mathrm{div}\, u^{\varepsilon,\alpha}\mathrm{d}x\\&\le\ \sum_{i=1}^3\|u_i^{\varepsilon,\alpha}\|_{L^3}\,\|\partial_i u_i^{\varepsilon,\alpha}\|_{L^3}\,\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^3}\\&\le\ 3K_2^3\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\,\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32}}\,\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\\&\le\ 9K_2^3\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\,\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32}},\end{aligned}$$

where K_2 is the constant such that

$$\|f\|_{L^3(\mathbb{R}^3)} \leq K_2 \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Now, assume that

$$\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{36K_2^3}. \tag{3.31}$$

Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{\frac{1}{2}}_{\varepsilon,\alpha}(0) \leq \left(-\frac{3}{4} + 9K_2^3 \|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}}\right) \|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}} < -\frac{1}{4} \|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{3}{2}}}$$

and $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ therefore decreases on an interval $[0,\tau]$. Set

$$\tau = \sup \left\{ t < T \ : \ E^{\frac{1}{2}}_{\varepsilon,\alpha} \text{ decreases on } [0,t] \right\}.$$

Since $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ decreases on $[0,\tau]$, we have

$$\|u^{\varepsilon,\alpha}(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} \leq 2E^{\frac{1}{2}}_{\varepsilon,\alpha}(\tau) \leq E^{\frac{1}{2}}_{\varepsilon,\alpha}(0) \leq 4\|u_0^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}}} \leq \left(\frac{1}{18K_2^3}\right)^2$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{\frac{1}{2}}_{\varepsilon,\alpha}(\tau) \leq -\frac{1}{4} \|u^{\varepsilon,\alpha}(\tau)\|^2_{\dot{H}^{\frac{3}{2}}}.$$

Thus $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ decreases on $[0,\tau']$, where $\tau' > \tau$. We have proved by contradiction that the energy decays on the whole interval $[0,T)$.

Now, we can prove the following lemma :

Lemma 3.6. *Assume that* $\|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} < \dfrac{1}{36K_2^3}$ *and*

$$(H') \qquad i)\ \varepsilon^{\frac{1+\delta}{2}} \|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{3}{2}+\delta}} = o(1)\ , \qquad ii)\ \varepsilon^{\frac{\delta}{2}} \|u_0^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}+\delta}} = o(1). \tag{3.32}$$

when ε *goes to zero.*
Now, recall that $0 \leq T \leq T^{\max}$ *is defined by*

$$T = \sup \left\{ 0 \leq \tau \leq T^{\max} \ : \ \forall\, t \in [0,\tau),\ \|u^{\varepsilon,\alpha}(t)\|_{L^\infty} < \frac{1}{2K_1\sqrt{\varepsilon}} \right\}.$$

Then $T > 0$ *and there exists a large number* N*, depending only on* δ*, and a constant* $C > 1$ *such that*

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq C^N E_\varepsilon^{\frac{1}{2}+\delta}(0) \tag{3.33}$$

for all $t \in [0,T)$ *and* ε *small enough.*

Proof. First, let us compute the time derivative of this energy :

$$\begin{aligned}
\frac{dE^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}}{dt}(t) &= \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}+\delta}} - \|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x \\
&\quad -\varepsilon\|\partial_t u^{\varepsilon,\alpha} + (u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}+\delta}} - \frac{1}{\alpha}\|\operatorname{div} u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}+\delta}} \\
&\le \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}+\delta}} - \|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x \\
&\le \left(K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty} - 1\right)\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} - \int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x.
\end{aligned}$$

On the time interval $[0,T)$, we have $K_1^2\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^\infty} - 1 \le -\frac{1}{2}$ and we can estimate the integral on the right hand side by

$$\begin{aligned}
\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\delta}(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}.\Lambda^{\frac{1}{2}+\delta}u^{\varepsilon,\alpha}\,\mathrm{d}x &\le \left\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{1}{2}+\delta}}\left\|u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{1}{2}+\delta}} \\
&\le K_1\|u^{\varepsilon,\alpha}\|_{L^\infty}\left\|u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{3}{2}+\delta}}\left\|u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{1}{2}+\delta}}
\end{aligned}$$

due to Lemma 3.1. Using that followed by (3.14), we obtain

$$\begin{aligned}
\frac{dE^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}}{dt}(t) &\le -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} + K_1\|u^{\varepsilon,\alpha}\|_{L^\infty}\left\|u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{3}{2}+\delta}}\left\|u^{\varepsilon,\alpha}\right\|_{\dot H^{\frac{1}{2}+\delta}} \\
&\le -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} + C\|u^{\varepsilon,\alpha}\|^{1+\delta}_{\dot H^{\frac{1}{2}+\delta}}\|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot H^{\frac{3}{2}+\delta}} \\
&\le -\frac{1}{2}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}+\delta}} + C\|u^{\varepsilon,\alpha}\|^{1-\delta}_{\dot H^{\frac{1}{2}}}\|u^{\varepsilon,\alpha}\|^{\delta}_{\dot H^{\frac{3}{2}}}\|u^{\varepsilon,\alpha}\|^{\delta}_{\dot H^{\frac{1}{2}+\delta}}\|u^{\varepsilon,\alpha}\|^{2-\delta}_{\dot H^{\frac{3}{2}+\delta}} \\
&\le C_\delta\|u^{\varepsilon,\alpha}\|^{2\frac{1-\delta}{\delta}}_{\dot H^{\frac{1}{2}}}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}}}E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}(t),
\end{aligned}$$

where we have used standard interpolations and a Young inequality. Now, since we do not know if the energy $E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}$ decays, we will use that $E^{\frac{1}{2}}_{\varepsilon,\alpha}$ satisfies the inequality

$$\frac{dE^{\frac{1}{2}}_{\varepsilon,\alpha}}{dt}(t) \le -\frac{1}{4}\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{3}{2}}}$$

on the interval $[0,T)$.

Now, as in the 2D case, let us define the functional $\mathcal{E}^{\frac{1}{2}}_{\varepsilon,\alpha,N} := E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right)^N$. Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{\frac{1}{2}}_{\varepsilon,\alpha,N}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right)^N + NE^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}\frac{\mathrm{d}}{\mathrm{d}t}E^{\frac{1}{2}}_{\varepsilon,\alpha}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right)^{N-1} \\
&\le E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right)^{N-1}\|u^{\varepsilon,\alpha}(t)\|^2_{\dot H^{\frac{3}{2}}}\left[C_\delta\|u^{\varepsilon,\alpha}(t)\|^{2\frac{1-\delta}{\delta}}_{\dot H^{\frac{1}{2}}}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right) - \frac{N}{4}\right] \\
&\le E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}\left(1+E^{\frac{1}{2}}_{\varepsilon,\alpha}\right)^{N-1}\|u^{\varepsilon,\alpha}(t)\|^2_{\dot H^{\frac{3}{2}}}\left[C_\delta\|u_0^{\varepsilon,\alpha}\|^{2\frac{1-\delta}{\delta}}_{\dot H^{\frac{1}{2}}}\left(1+2\|u_0^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}}}\right) - \frac{N}{4}\right].
\end{aligned}$$

Taking $N = N(\delta)$ large enough, we obtain that $\mathcal{E}^{\frac{1}{2}}_{\varepsilon,\alpha,N}$ decays and we have

$$E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}(t) \leq E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}(0)\left(1 + E^{\frac{1}{2}}_{\varepsilon,\alpha}(0)\right)^N.$$

Finally, notice that

$$E^{\frac{1}{2}}_{\varepsilon,\alpha}(0) \leq 2\|u_0^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}}},$$

so the proof of the lemma is finished. □

As in the 2D case, the smallness assumptions on the initial data yield the boundedness of $E^{\frac{1}{2}+\delta}_{\varepsilon,\alpha}$ for α and ε small enough. From this, we deduce that the solutions to $(HNS^{\varepsilon,\alpha})$ are global.

3.6.3 Convergence

Let $u^{\varepsilon,\alpha}$ and u^ε be the solutions to $(HNS^{\varepsilon,\alpha})$ and (HNS^ε) respectively with the same initial data

$$(u_0^{\varepsilon,\alpha}, u_1^{\varepsilon,\alpha}) = (u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3.$$

In order to prove that $u^{\varepsilon,\alpha}$ converges to u^ε in the $L^\infty_T \dot H^{\frac{1}{2}}$ norm, as α goes to 0, we shall use, as in the 2D case, a variant of the Dafermos modulated energy :

$$\begin{aligned}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon} &= \frac{1}{2}\|u^{\varepsilon,\alpha} - u^\varepsilon + \varepsilon\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|^2_{\dot H^{\frac{1}{2}}} + \frac{\varepsilon^2}{2}\|\partial_t(u^{\varepsilon,\alpha} - u^\varepsilon)\|^2_{\dot H^{\frac{1}{2}}}\\ &\quad + \varepsilon\|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{\dot H^{\frac{3}{2}}} + \frac{\varepsilon}{\alpha}\|\operatorname{div} u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac{1}{2}}}.\end{aligned}$$

As done in section 3.5.3, we shall compute the time derivative of $\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}$ and use equations $(HNS^{\varepsilon,\alpha})$ and (HNS^ε). Doing so, we obtain the following estimate :

$$\begin{aligned}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T) &\leq \mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(0) + \varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} - (u^\varepsilon.\nabla)u^\varepsilon\|^2_{L^2_T\dot H^{\frac{1}{2}}} - \|u^{\varepsilon,\alpha} - u^\varepsilon\|^2_{L^2_T\dot H^{\frac{3}{2}}} -\\ &\quad - \int_0^T\int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}p^\varepsilon.\Lambda^{\frac{1}{2}}\operatorname{div} u^{\varepsilon,\alpha}\,\mathrm{d}x\mathrm{d}t + 2\varepsilon\int_0^T\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}}\partial_t p^\varepsilon.\Lambda^{\frac{1}{2}}\operatorname{div} u^{\varepsilon,\alpha}\,\mathrm{d}x\mathrm{d}t +\\ &\quad + \int_0^T\int_{\mathbb{R}^3}\left[(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha} - (u^\varepsilon.\nabla)u^\varepsilon\right].\Lambda(u^{\varepsilon,\alpha} - u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\alpha}\|\operatorname{div} u^{\varepsilon,\alpha}\|^2_{L^2_T\dot H^{\frac{1}{2}}}.\end{aligned}$$

3.6.3.1 Estimate on $\varepsilon\|(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^{\varepsilon}.\nabla)u^{\varepsilon}\|^2_{L^2_T\dot H^{\frac12}}$

First, $A:=(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^{\varepsilon}.\nabla)u^{\varepsilon}$ writes

$$(u^{\varepsilon,\alpha}.\nabla)u^{\varepsilon,\alpha}-(u^{\varepsilon}.\nabla)u^{\varepsilon}=\big((u^{\varepsilon,\alpha}-u^{\varepsilon}).\nabla\big)u^{\varepsilon}+\big(u^{\varepsilon,\alpha}.\nabla\big)(u^{\varepsilon,\alpha}-u^{\varepsilon}).$$

Let us estimate the RHS using the dyadic Littlewood-Paley decomposition. First, by Lemma B.1 in the appendix, we have

$$\begin{aligned}\|(u^{\varepsilon,\alpha}-u^{\varepsilon}).\nabla u^{\varepsilon}\|^2_{\dot H^{\frac12}} &\le \sum_{i,j=1}^{3}\sum_{p\in\mathbb{Z}}2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\,.\,S_{p+1}(\partial_i u_j^{\varepsilon})\|^2_{L^2}\\ &+\sum_{i,j=1}^{3}\sum_{q\in\mathbb{Z}}2^q\|\Delta_q(\partial_i u_j^{\varepsilon})\,.\,S_q(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\|^2_{L^2}.\end{aligned}$$

The first term estimates easily :

$$\begin{aligned}\sum_p 2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\,.\,S_{p+1}(\partial_i u_j^{\varepsilon})\|^2_{L^2} &\le \sum_p 2^p\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\|^2_{L^2}\,\|S_{p+1}(\partial_i u_j^{\varepsilon})\|^2_{L^\infty}\\ &\le \|u^{\varepsilon}\|^2_{L^\infty}\sum_p 2^{3p}\,\|\Delta_p(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\|^2_{L^2}\\ &\le \|u^{\varepsilon}\|^2_{L^\infty}\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^2_{\dot H^{\frac32}}\end{aligned}$$

due to property (B.1) in the appendix. Now, notice that Sobolev embeddings imply that $u^{\varepsilon}\in\dot H^{\frac32+\delta}\subset W^{\frac32,\alpha}(\mathbb{R}^3)$, where $\alpha=\frac{6}{3-2\delta}$. Then, if $\bar\alpha=\frac{3}{\delta}$, we have

$$\begin{aligned}\sum_q 2^q\|\Delta_q(\partial_i u_j^{\varepsilon})\,.\,S_q(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\|^2_{L^2} &\le \sum_q 2^q\|\Delta_q(\partial_i u_j^{\varepsilon})\|^2_{L^\alpha}\,\|S_q(u_i^{\varepsilon,\alpha}-u_i^{\varepsilon})\|^2_{L^{\bar\alpha}}\\ &\le \|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^2_{L^{\bar\alpha}}\sum_q 2^{q+3q\left(1-\frac{2}{\alpha}\right)}\|\Delta_q(\partial_i u_j^{\varepsilon})\|^2_{L^2}\\ &\le C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^2_{\dot H^{\frac32-\delta}}\sum_q 2^{3q+3q\left(1-\frac{2}{\alpha}\right)}\|\Delta_q u_j^{\varepsilon}\|^2_{L^2}\\ &= C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^2_{\dot H^{\frac32-\delta}}\sum_q 2^{6q-\frac{6q}{\alpha}}\|\Delta_q u_j^{\varepsilon}\|^2_{L^2}\\ &= C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^2_{\dot H^{\frac32-\delta}}\sum_q 2^{2q\left(\frac32+\delta\right)}\|\Delta_q u_j^{\varepsilon}\|^2_{L^2}\\ &\le C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^{\delta}_{\dot H^{\frac12}}\,\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|^{1-\delta}_{\dot H^{\frac32}}\|u^{\varepsilon}\|_{\dot H^{\frac32+\delta}}\\ &\le C_\delta\left(\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac12}}+\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac32}}\right)\|u^{\varepsilon}\|_{\dot H^{\frac32+\delta}}.\end{aligned}$$

Now, let us estimate

$$\begin{aligned}\|u^{\varepsilon,\alpha}.\nabla(u^{\varepsilon,\alpha}-u^{\varepsilon})\|_{\dot H^{\frac12}} &\le \sum_{i,j=1}^{3}\sum_{p\in\mathbb{Z}} 2^p\|\Delta_p u_i^{\varepsilon,\alpha}\,.\,S_{p+1}\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\|_{L^2}^2\\ &+ \sum_{i,j=1}^{3}\sum_{q\in\mathbb{Z}} 2^q\|\Delta_q\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\,.\,S_q u_i^{\varepsilon,\alpha}\|_{L^2}^2.\end{aligned}$$

We know that $u^{\varepsilon,\alpha}\in \dot H^{\frac32+\delta}\subset \dot W^{\frac32,\alpha}$, where $\alpha=\frac{6}{3-2\delta}$. Let $\bar\alpha=\frac{3}{\delta}$. Then

$$\begin{aligned}\sum_p 2^p\|\Delta_p u_i^{\varepsilon,\alpha}\,.\,S_{p+1}\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\|_{L^2}^2 &\le \sum_p 2^p\|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2\|S_{p+1}\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\|_{L^{\bar\alpha}}^2\\ &\le \sum_p 2^{3p}\|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2\|S_{p+1}(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\|_{L^{\bar\alpha}}^2\\ &\le \|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{L^{\bar\alpha}}^2\sum_p 2^{3p}\|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^\alpha}^2\\ &\le \|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{L^{\bar\alpha}}^2\sum_p 2^{3p+6p\left(\frac12-\frac1\alpha\right)}\|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^2}^2\\ &= \|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{L^{\bar\alpha}}^2\sum_p 2^{2p\left(\frac32+\delta\right)}\|\Delta_p u_i^{\varepsilon,\alpha}\|_{L^2}^2\\ &\le C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac32-\delta}}^2\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32+\delta}}^2\end{aligned}$$

since $\dot H^{\frac32-\delta}\subset L^{\bar\alpha}(\mathbb{R}^3)$. So, by interpolation and a Young inequality, we get

$$\sum_p 2^p\|\Delta_p u_i^{\varepsilon,\alpha}.S_{p+1}\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\|_{L^2}^2\le C_\delta\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32+\delta}}\left(\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac12}}+\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac32}}\right).$$

Immediately, the remaining term estimates

$$\sum_q 2^q\|\Delta_q\partial_i(u_j^{\varepsilon,\alpha}-u_j^{\varepsilon})\,.\,S_q u_i^{\varepsilon,\alpha}\|_{L^2}^2\le C\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{\dot H^{\frac32}}^2\|u^{\varepsilon,\alpha}\|_{L^\infty}^2. \tag{3.34}$$

Finally, we obtain

$$\begin{aligned}\varepsilon\|A\|_{L^2_T\dot H^{\frac12}}^2 &\le C_{\delta,\eta}\varepsilon\int_0^T\left(\|u^{\varepsilon,\alpha}\|_{\dot H^{\frac32+\delta}}^2+\|u^{\varepsilon}\|_{\dot H^{\frac32+\delta}}^2\right)\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t+\\ &+ C\varepsilon\left(\|u^{\varepsilon,\alpha}\|_{L^\infty}^2+\|u^{\varepsilon}\|_{L^\infty}^2\right)\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{L^2_T\dot H^{\frac32}}^2+\\ &+ C_\delta\eta\varepsilon\left(\|u^{\varepsilon,\alpha}\|_{L^\infty_T\dot H^{\frac32+\delta}}^2+\|u^{\varepsilon}\|_{L^\infty_T\dot H^{\frac32+\delta}}^2\right)\|u^{\varepsilon,\alpha}-u^{\varepsilon}\|_{L^2_T\dot H^{\frac32}}^2,\end{aligned}$$

where we use that $\varepsilon\|u^{\varepsilon,\alpha}\|_{L^\infty_T\dot H^{\frac32+\delta}}^2=\mathcal{O}(1)$ (see subsection 3.6.1).

3.6.3.2 Estimate on $\int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \nabla p^\varepsilon . \Lambda^{\frac{1}{2}} u^{\varepsilon,\alpha} \, \mathbf{d}x \, \mathbf{d}t$

Applying the div operator to (HNS^ε), we obtain the identity

$$p^\varepsilon = -\frac{1}{\Delta} \mathrm{div}\,(u^\varepsilon.\nabla) u^\varepsilon.$$

Then, recall that $\mathrm{div}\, u^{\varepsilon,\alpha} \in L^2_T \dot{H}^{\frac{1}{2}}$ and $\|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T \dot{H}^{\frac{1}{2}}} = \mathcal{O}(\sqrt{\alpha})$. Now, we easily estimate the integral as follows :

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \nabla p^\varepsilon . \Lambda^{\frac{1}{2}} u^{\varepsilon,\alpha} \, \mathrm{d}x \, \mathrm{d}t \right| &= \left| \int_0^T \int_{\mathbb{R}^3} \Lambda p^\varepsilon . \mathrm{div}\, u^{\varepsilon,\alpha} \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\leq \|\Lambda p^\varepsilon\|_{L^2_T L^{\frac{3}{2}}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \sum_{i,j,k=1}^{3} \|\big(\partial_k u_i^\varepsilon\big) u_j^\varepsilon\|_{L^2_T L^{\frac{3}{2}}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \sum_{i,j,k=1}^{3} \|\partial_k u_i^\varepsilon\|_{L^2_T L^3} \|u_j^\varepsilon\|_{L^\infty_T L^3} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T L^3} \\
&\leq C \|u^\varepsilon\|_{L^2_T \dot{H}^{\frac{3}{2}}} \|u^\varepsilon\|_{L^\infty_T \dot{H}^{\frac{1}{2}}} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^2_T \dot{H}^{\frac{1}{2}}} \\
&= \mathcal{O}(\sqrt{\alpha}).
\end{aligned}$$

3.6.3.3 Estimate on $2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \nabla p^\varepsilon . \Lambda^{\frac{1}{2}} \partial_t u^{\varepsilon,\alpha} \, \mathbf{d}x \, \mathbf{d}t$

First, two integrations by parts (one in space and another in time) give

$$I := 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \nabla p^\varepsilon . \Lambda^{\frac{1}{2}} \partial_t u^{\varepsilon,\alpha} \, \mathrm{d}x \, \mathrm{d}t = 2\varepsilon \int_0^T \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}} \partial_t p^\varepsilon . \Lambda^{\frac{1}{2}} \mathrm{div}\, u^{\varepsilon,\alpha} \, \mathrm{d}x \, \mathrm{d}t$$

which is easier to estimate.
Notice that $\Lambda^{\frac{1}{2}} \mathrm{div}\, u^{\varepsilon,\alpha} \in L^2_T L^2(\mathbb{R}^3)$. Now, recall that

$$\Lambda^{\frac{1}{2}} \partial_t p^\varepsilon = \Lambda^{\frac{1}{2}} \partial_t \frac{1}{\Delta} \mathrm{div}\,(u^\varepsilon.\nabla) u^\varepsilon.$$

So we have

$$\begin{aligned}
\|\Lambda^{\frac{1}{2}} \partial_t p^\varepsilon\|_{L^2_T L^2} &\leq C \|\partial_t (u^\varepsilon.\nabla) u^\varepsilon\|_{L^2_T \dot{H}^{-\frac{1}{2}}} \\
&= C \|\partial_t \big(u^\varepsilon \otimes u^\varepsilon\big)\|_{L^2_T \dot{H}^{\frac{1}{2}}} \\
&\leq C \|\partial_t u^\varepsilon \otimes u^\varepsilon\|_{L^2_T \dot{H}^{\frac{1}{2}}}.
\end{aligned}$$

We shall estimate this term by a dyadic Littlewood-Paley decomposition. Using Lemma B.1 in the appendix, we have

$$\|u^\varepsilon \otimes \partial_t u^\varepsilon\|^2_{\dot H^{\frac12}} \leq \sum_{i,j}\sum_{p\in\mathbb{Z}} 2^p \|\Delta_p u_i^\varepsilon\, S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^2} + \sum_{i,j}\sum_{q\in\mathbb{Z}} 2^q\|\Delta_q \partial_t u_j^\varepsilon\, S_q u_i^\varepsilon\|^2_{L^2}.$$

Then we estimate each term separately, using Bernstein inequalities. First, we have

$$\begin{aligned}\sum_p 2^p\|\Delta_p u_i^\varepsilon\, S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^2} &\leq \sum_p 2^p \|\Delta_p u_i^\varepsilon\|^2_{L^2}\|S_{p+1}\partial_t u_j^\varepsilon\|^2_{L^\infty}\\ &\leq C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\sum_p 2^{-2p}\left\|\sum_{-1\leq k\leq p}\Delta_k\partial_t u_j^\varepsilon\right\|^2_{L^\infty}\\ &\leq C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\sum_p 2^{-2p}\sum_{-1\leq k\leq p} 2^{3k}\left\|\Delta_k\partial_t u_j^\varepsilon\right\|^2_{L^2}\\ &\leq C\|u^\varepsilon\|^2_{\dot H^{\frac32}}\|\partial_t u^\varepsilon\|^2_{\dot H^{\frac12}}.\end{aligned}$$

Similarly, we obtain that

$$\sum_q 2^q\|\Delta_q\partial_t u_j^\varepsilon\, S_q u_i^\varepsilon\|^2_{L^2} \leq \sum_q 2^q\|\Delta_q\partial_t u_j^\varepsilon\|^2_{L^2}\|S_q u_i^\varepsilon\|^2_{L^\infty} \leq C\|\partial_t u^\varepsilon\|^2_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}}.$$

So finally, we have

$$\begin{aligned} I &\leq C\varepsilon\left(\|u^\varepsilon\|_{L^\infty_T\dot H^{\frac32}}\left\|\partial_t u_j^\varepsilon\right\|_{L^2_T\dot H^{\frac12}} + \|\partial_t u^\varepsilon\|_{L^\infty_T\dot H^{\frac12+\delta}}\|u^\varepsilon\|_{L^2_T\dot H^{\frac32+\delta}}\right)\|\operatorname{div} u^{\varepsilon,\alpha}\|_{L^2_T\dot H^{\frac12}}\\ &= \mathcal{O}\left(\sqrt{\alpha}\right).\end{aligned}$$

3.6.3.4 Estimate on $\int_0^T\int_{\mathbb{R}^3}\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).\left(u^\varepsilon.\nabla u^\varepsilon - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha}\right)\,\mathbf{d}x\,\mathbf{d}t$

Let us write $u^\varepsilon.\nabla u^\varepsilon - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha} = (u^\varepsilon - u^{\varepsilon,\alpha}).\nabla u^\varepsilon + u^{\varepsilon,\alpha}.\nabla(u^\varepsilon - u^{\varepsilon,\alpha})$. First, using the Sobolev embeddings $\dot H^1 \subset L^6(\mathbb{R}^3)$ and $\dot H^{\frac12}\subset L^3(\mathbb{R}^3)$, we estimate the integral

$$\tilde A := \int_0^T\int_{\mathbb{R}^3}\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).(u^\varepsilon - u^{\varepsilon,\alpha}).\nabla u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

as follows :

$$
\begin{aligned}
\tilde{A} &\leq \int_0^T \|\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon)\|_{L^2}\,\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{L^6}\,\|\nabla u^\varepsilon\|_{L^3}\,\mathrm{d}t\\
&\leq C\int_0^T \|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^1}\,\|u^\varepsilon\|_{\dot H^{\frac32}}\,\mathrm{d}t\\
&\leq C\int_0^T \|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac12}}\,\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac32}}\,\|u^\varepsilon\|_{\dot H^{\frac32}}\,\mathrm{d}t\\
&\leq \frac{C}{\eta}\int_0^T \|u^\varepsilon\|^2_{\dot H^{\frac32}}\,\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\,\mathrm{d}t+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{L^2_T\dot H^{\frac32}}
\end{aligned}
$$

Now, we are left with the term :

$$
\begin{aligned}
\Lambda(u^{\varepsilon,\alpha}-u^\varepsilon).u^{\varepsilon,\alpha}.\nabla(u^\varepsilon-u^{\varepsilon,\alpha}) &= \sum_{i,j=1}^3 \partial_j(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)+\\
&\quad+\sum_{i,j=1}^3 \partial_j(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\,u_i^\varepsilon\,\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)=I_1+I_2.
\end{aligned}
$$

One can easily check that the first part estimates as follows :

$$
\begin{aligned}
\int_{\mathbb{R}^3} I_1\,\mathrm{d}x &= \frac12\sum_{i=1}^3\int_{\mathbb{R}^3}\partial_i(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)^2\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x\\
&\leq C\sum_{i=1}^3\|u_i^{\varepsilon,\alpha}-u_i^\varepsilon\|_{L^3}\|\partial_i(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\|_{L^3}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{L^3}\\
&\leq C\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac12}}\|u^{\varepsilon,\alpha}-u^\varepsilon\|_{\dot H^{\frac32}}\|\mathrm{div}\,u^{\varepsilon,\alpha}\|_{\dot H^{\frac12}}\\
&\leq \frac{C\alpha}{\eta}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}+\eta\|u^{\varepsilon,\alpha}-u^\varepsilon\|^2_{\dot H^{\frac32}},
\end{aligned}
$$

where η is small. Besides, integrating by parts the second term, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} I_2\,\mathrm{d}x &= -\sum_{i,j=1}^3\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)\partial_j u_i^\varepsilon\partial_i(u_j^{\varepsilon,\alpha}-u_j^\varepsilon)\,\mathrm{d}x\\
&\quad-\sum_{i=1}^3\int_{\mathbb{R}^3}(u_i^{\varepsilon,\alpha}-u_i^\varepsilon)u_i\partial_i\mathrm{div}\,u^{\varepsilon,\alpha}\,\mathrm{d}x.
\end{aligned}
$$

A second integration by parts yields

$$\begin{aligned}
\int_{\mathbb{R}^3} I_2 \, \mathrm{d}x &= -\sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^{\varepsilon}) \partial_j u_i^{\varepsilon} \partial_i (u_j^{\varepsilon,\alpha} - u_j^{\varepsilon}) \, \mathrm{d}x \\
&\quad + \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i (u_i^{\varepsilon,\alpha} - u_i^{\varepsilon}) \, u_i^{\varepsilon} \, \mathrm{div}\, u^{\varepsilon,\alpha} \, \mathrm{d}x \\
&\quad + \sum_{i=1}^{3} \int_{\mathbb{R}^3} (u_i^{\varepsilon,\alpha} - u_i^{\varepsilon}) \, \partial_i u_i^{\varepsilon} \, \mathrm{div}\, u^{\varepsilon,\alpha} \, \mathrm{d}x \\
&= I + II + III
\end{aligned}$$

We estimate these three terms as follows :

$$\begin{aligned}
I &\le \sum_{i,j=1}^{3} \|u_i^{\varepsilon,\alpha} - u_i^{\varepsilon}\|_{L^3} \, \|\partial_j u_i^{\varepsilon}\|_{L^3} \, \|\partial_i (u_j^{\varepsilon,\alpha} - u_j^{\varepsilon})\|_{L^3} \\
&\le C \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \, \|u^{\varepsilon}\|_{\dot{H}^{\frac{3}{2}}} \, \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^{\frac{3}{2}}} \\
&\le \frac{C}{\eta} \|u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}} \, \mathcal{E}_{\varepsilon,\alpha,u^{\varepsilon}} \; + \; \eta \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}}.
\end{aligned}$$

$$\begin{aligned}
II &\le \sum_{i=1}^{3} \|\partial_i (u_i^{\varepsilon,\alpha} - u_i^{\varepsilon})\|_{L^3} \, \|u_i^{\varepsilon}\|_{L^3} \, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^3} \\
&\le C \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^{\frac{3}{2}}} \, \|u^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} \\
&\le C \|u_0^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \, \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|_{\dot{H}^{\frac{3}{2}}} \, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{\dot{H}^{\frac{1}{2}}} \\
&\le \eta \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}} + \frac{C}{\eta} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}}}.
\end{aligned}$$

$$\begin{aligned}
III &\le \sum_{i=1}^{3} \|u_i^{\varepsilon,\alpha} - u_i^{\varepsilon}\|_{L^3} \, \|\partial_i u_i^{\varepsilon}\|_{L^3} \, \|\mathrm{div}\, u^{\varepsilon,\alpha}\|_{L^3} \\
&\le C \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|^2_{\dot{H}^{\frac{1}{2}}} \, \|u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}} \; + \; \frac{1}{2} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}}} \\
&\le C \|u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}} \, \mathcal{E}_{\varepsilon,\alpha,u^{\varepsilon}} \; + \; \frac{1}{2} \|\mathrm{div}\, u^{\varepsilon,\alpha}\|^2_{\dot{H}^{\frac{1}{2}}}.
\end{aligned}$$

Summarizing, we have obtained that

$$\begin{aligned}
\int_0^T \!\! \int_{\mathbb{R}^3} \Lambda (u^{\varepsilon,\alpha} - u^{\varepsilon}). \, (u^{\varepsilon}.\nabla u^{\varepsilon} - u^{\varepsilon,\alpha}.\nabla u^{\varepsilon,\alpha}) \; \mathrm{d}x \, \mathrm{d}t \; &\le \int_0^T \Big(C_\eta \|u^{\varepsilon}\|^2_{\dot{H}^{\frac{3}{2}}} + C_\eta \alpha \Big) \, \mathcal{E}_{\varepsilon,\alpha,u^{\varepsilon}}(t) \mathrm{d}t \, + \\
&\quad + C\eta \|u^{\varepsilon,\alpha} - u^{\varepsilon}\|^2_{L^2_T \dot{H}^{\frac{3}{2}}} + \mathcal{O}(\alpha).
\end{aligned}$$

3.6.3.5 Conclusion

Since $\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(0) = 0$, if we choose η and ε small enough, we obtain the estimate

$$\begin{aligned}\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T) \ &\leq \ \int_0^T \Big[C_{\delta,\eta}\varepsilon\left(\|u^{\varepsilon,\alpha}\|^2_{\dot H^{\frac32+\delta}} + \|u^\varepsilon\|^2_{\dot H^{\frac32+\delta}}\right) + C_\eta\|u^\varepsilon\|^2_{\dot H^{\frac32}} + C_\eta\alpha\Big]\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(t)\mathrm{d}t \\ &\quad + \mathcal{O}(\sqrt{\alpha}).\end{aligned}$$

Now, notice that $u^\varepsilon \in L^2_T\dot H^{\frac32} \cap L^2_T\dot H^{\frac32+\delta}$ and that $\varepsilon\|u^{\varepsilon,\alpha}\|^2_{L^2_T\dot H^{\frac32+\delta}} = \mathcal{O}(1)$ then apply the Gronwall's lemma and obtain that, for all positive T,

$$\mathcal{E}_{\varepsilon,\alpha,u^\varepsilon}(T) = \mathcal{O}(\sqrt{\alpha}).$$

As in the 2D case, Theorem 3.1 concludes the proof of Theorem 3.2.

Chapitre 4

Perspectives

Dans un premier temps après ma thèse, je voudrais continuer à travailler sur la perturbation hyperbolique (HNS^{ε}) pour obtenir un résultat d'existence globale et de convergence qui réunit les avantages des Théorèmes 1.4 (voir [35]) et 1.7 (voir [20]). Nous voudrions que les données initiales soient peu régulières : $H^1 \times L^2(\mathbb{R}^2)^2$ (l'estimation de Strichartz utilisée par Paicu et Raugel autorise même moins de régularité) et que les hypothèses sur la taille des données soient les moins restrictives possibles. De même, nous cherchons à montrer un résultat similaire en dimension 3. Pour cela, nous combinerons notre méthode d'énergie et l'estimation de Strichartz sur les hautes fréquences de Paicu et Raugel. Ce travail fera l'objet d'une collaboration avec Geneviève Raugel.

De plus, l'importance des simulations numériques dans le domaine de la mécanique des fluides étant avérée, je voudrais mieux comprendre, du point de vue numérique, les approximations de (NS) considérées dans cette thèse. Concernant l'approximation à faible compressibilité, $(HNS^{\varepsilon,\alpha})$, certains pensent qu'elle ne donnerait pas une bonne approximation numérique des solutions de (NS) [6], néanmoins ce modèle n'a pas encore été testé à ce jour.

Nous avons remarqué que les approximations considérées dans cette thèse ne respectent pas l'invariance galiléenne. Ce ne sont donc pas des équations physiques. Pour les rendre *admissibles*, il suffirait de considérer la dérivée matérielle $D_t = \partial_t + (v.\nabla)$ au lieu de ∂_t dans la partie que l'on rajoute à (NS) (voir [12, 33]). On obtient alors les équations

$$(GHNS^{\varepsilon}) \quad \left\{ \begin{array}{rcl} \varepsilon D_t^2 v^{\varepsilon} + D_t v^{\varepsilon} & = & \Delta v^{\varepsilon} - \nabla p^{\varepsilon} \\ \operatorname{div} v^{\varepsilon} & = & 0 \end{array} \right. ,$$

$$(GHNS^{\varepsilon,\alpha}) \quad \varepsilon D_t^2 v^{\varepsilon,\alpha} + D_t v^{\varepsilon,\alpha} = \Delta v^{\varepsilon,\alpha} + \frac{1}{\alpha}\nabla \mathrm{div}\, v^{\varepsilon,\alpha}\,.$$

Il faudrait alors montrer que ces équations sont encore des approximations de (NS) et que la deuxième équation a une vitesse de propagation finie.

Pour finir, il est classique de s'intéresser aux équations de la MHD puisqu'elles font intervenir les équations de Navier-Stokes en les couplant avec d'autres systèmes comme ici, le système de Maxwell-Navier-Stokes

$$(MNS) \quad \left\{ \begin{array}{rcl} \partial_t v - \nu\Delta v &=& -(v.\nabla)v - \nabla p + j \times B \\ \partial_t E - \mathrm{curl} B &=& -j \\ \partial_t B + \mathrm{curl} E &=& 0 \\ \mathrm{div}\, v = \mathrm{div}\, B &=& 0 \\ \sigma\,(E + v \times B) &=& j \end{array} \right. ,$$

où $v : \mathbb{R}^+ \times \mathbb{R}^n$ est la vitesse du fluide considéré, E le champ électrique et B le champ magnétique. Le courant électrique j est donnée par la loi d'Ohm $\sigma\,(E + v \times B) = j$, où la constante σ représente la conductivité. Dans leurs articles [32] et [18], Germain et Masmoudi montrent que le système (MNS) est bien posé en utilisant des estimées dispersives, des estimées locales et des méthodes d' énergie. Ayant utilisé le même genre d'outils dans ma thèse, les équations de la MHD me paraissent correspondre à ce que je voudrais faire.

Annexe A

Estimation de Strichartz pour l'équation d'ondes amorties

Dans cette annexe, nous démontrons l'inégalité de Strichartz utilisée par Paicu et Raugel dans [36]. Soit u la solution du problème de Cauchy

$$\left\{\begin{array}{rcl} \partial_{tt}u + \partial_t u - \Delta u & = & G = G_2 + G_3 + G_4 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1) \end{array}\right. .$$

Soit $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ telle que $\varphi(\xi) = 1$ si $|\xi| \leq 1$ et soient $\delta_0, \cdots, \delta_4 > 0$ et $1 < p < 2$. Nous allons démontrer l'estimation suivante :

Théorème A.1 (Inégalité de Strichartz)**.** *Si on note* $S_0 f = \mathcal{F}^{-1}\left(\varphi(\xi)\hat{f}(\xi)\right)$, *alors il existe une constante* $C_0 > 0$ *telle que*

$$\begin{array}{rcl} \|(I - S_0)u\|_{L^2L^\infty} & \leq & C_0\big(\|u_0\|_{\dot{H}^{1+\delta_0}} + \|u_1\|_{\dot{H}^{\delta_1}}\big) \\ & & +C_0\big(\|G_2\|_{L^1\dot{H}^{\delta_2}} + \|G_3\|_{L^2\dot{H}^{\frac{1}{2}+\delta_3}} + \|G_4\|_{L^p\dot{H}^{1-\frac{1}{p}+\delta_4}}\big) \end{array}$$

Pour commencer, nous allons exprimer $(I - S_0)u$ en fonction des données initiales u_0 et u_1 et du terme source G. Pour cela, on définit les opérateurs $U^+(t)$ et $U^-(t)$ par

$$U^\pm(t)f = \mathcal{F}^{-1}\left(e^{\pm i\frac{t}{2}\sqrt{4|\xi|^2-1}}(1 - \varphi(\xi))\hat{f}(\xi)\right).$$

Maintenant, soit ψ une fonction test vérifiant $\psi(\xi) = 1$ pour $|\xi| \leq \frac{1}{2}$ et

$(1-\psi)(1-\varphi) = 1-\varphi$. On définit alors les opérateurs

$$\begin{aligned}
A^{\pm} f &= \mathcal{F}^{-1}\left(\frac{1 \pm i\sqrt{4|\xi|^2-1}}{2i\sqrt{4|\xi|^2-1}}(1-\psi(\xi))\hat{f}(\xi)\right),\\
A(t) &= e^{-\frac{t}{2}}U^{+}(t)A^{+}f + e^{-\frac{t}{2}}U^{-}(t)A^{-}f\,,\\
B^0 f &= \mathcal{F}^{-1}\left(\frac{1}{i\sqrt{4|\xi|^2-1}}(1-\psi(\xi))\hat{f}(\xi)\right),\\
B(t) &= e^{-\frac{t}{2}}\big(U^{+}(t)-U^{-}(t)\big)\,B^0 f\,.
\end{aligned}$$

On peut maintenant écrire

$$(I-S_0)u = A(t)u_0 + B(t)u_1 + \int_0^t B(t-s)G(s)\,\mathrm{d}s.$$

Pour démontrer l'estimation du Théorème A.1, on se ramène à prouver le lemme suivant.

Lemme A.1. *Soit u_0 telle que $\widehat{u_0}(\xi) = 0$ pour $|\xi| \leq \frac{1}{2}$. Alors, pour $\delta > 0$, on a*

$$\|e^{-\frac{t}{2}}U^{+}(t)u_0\|_{L^2_t L^\infty} \leq C_\delta \|u_0\|_{\dot{H}^{1+\delta}}\,. \tag{A.1}$$

Démonstration. Pour démontrer (A.1), supposons que l'on ait l'estimation de dispersion

$$\begin{aligned}
\forall j \geq -1\,,\ \forall t > 0\,,\ \forall f \in L^1(\mathbb{R}^3)\,,\ \|U^{+}(t)\Delta_j f\|_{L^\infty} &\leq C2^{3j}\min\left(1, \frac{1}{2^j t}\right)\|f\|_{L^1}\\
&\leq C2^{(2+\varepsilon)j}t^{\varepsilon-1}\|f\|_{L^1}
\end{aligned}$$

pour $0 < \varepsilon \leq 1$. Nous la démontrerons dans le lemme suivant.
Soit $f \in L^2_t L^\infty(\mathbb{R}^3)$. On a

$$\begin{aligned}
\left\|\int_0^t e^{-\frac{t-s}{2}}U^{+}(t-s)\Delta_j f(s)\,\mathrm{d}s\right\|_{L^2_t L^\infty} &\leq C2^{(2+\varepsilon)j}\left\|\int_0^t e^{-\frac{t-s}{2}}\frac{1}{(t-s)^{1-\varepsilon}}\|f(s)\|_{L^1}\,\mathrm{d}s\right\|_{L^2_t}\\
&\leq C2^{(2+\varepsilon)j}\|f\|_{L^2_t L^1}.
\end{aligned}$$

En utilisant des espaces à poids en temps, on réécrit cette estimation de la façon suivante :

$$\left\|\int_0^t U^{+}(t-s)\Delta_j f(s)\,\mathrm{d}s\right\|_{L^2(e^{-t}\,\mathrm{d}t, L^\infty)} \leq C2^{(2+\varepsilon)j}\|f\|_{L^2(e^{t}\,\mathrm{d}t, L^1)}.$$

Soit $T > 0$. On a

$$\begin{aligned}
&\left\|\int_0^T U^+(T-s)\Delta_j f(s)\,\mathrm{d}s\right\|^2_{L^2(\mathbb{R}^3)}\\
&\quad = \int_{\mathbb{R}^3}\iint_{[0,T]^2} \overline{U^+(T-t)\Delta_j f(t,x)}\, U^+(T-s)\Delta_j f(s,x)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^3}\left(\iint_{(s,t)\in[0,T]^2, s\leq t}\cdots\,\mathrm{d}s\,\mathrm{d}t + \iint_{(s,t)\in[0,T]^2, s\geq t}\cdots\,\mathrm{d}s\,\mathrm{d}t\right)\mathrm{d}x\\
&\quad = 2\Re\int_{\mathbb{R}^3}\int_0^T \overline{U^+(T-t)\Delta_j f(t,x)}\int_0^t U^+(T-s)\Delta_j f(s,x)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

En notant $\sigma(\xi) = \sqrt{4|\xi|^2-1}$ et en écrivant $U^+(t) = e^{i\frac{t}{2}\sigma(D)}(1-\varphi(D))$, on obtient

$$\begin{aligned}
&\left\|\int_0^T U^+(T-s)\Delta_j f(s)\,\mathrm{d}s\right\|^2_{L^2(\mathbb{R}^3)}\\
&\quad = 2\Re\int_{\mathbb{R}^3}\int_0^T \overline{(1-\varphi(D))\Delta_j f(t,x)}\int_0^t U^+(t-s)\Delta_j f(s,x)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Par l'inégalité de Hölder, on obtient l'estimation

$$\sup_{T>0}\left\|\int_0^T U^+(T-s)\Delta_j f(s)\,\mathrm{d}s\right\|_{L^2(\mathbb{R}^3)} \leq C2^{(1+\frac{\varepsilon}{2})j}\|f\|_{L^2(e^t\,\mathrm{d}t, L^1)}. \tag{A.2}$$

Soient maintenant $u_0 \in L^2$, $f \in L^2(e^t\,\mathrm{d}t, L^1)$ et $T > 0$. On écrit alors

$$\int_0^T\int_{\mathbb{R}^3}\overline{U^+(t)\Delta_j u_0(x)}\Delta_j f(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^3}\overline{\Delta_j u_0(x)}\int_0^T U^+(T-s)\Delta_j f(T-s,x)\,\mathrm{d}s\,\mathrm{d}x$$

et on déduit de (A.2) par dualité que

$$\left\|e^{-\frac{t}{2}}U^+(t)\Delta_j u_0\right\|_{L^2_t L^\infty} \leq C2^{(1+\frac{\varepsilon}{2})j}\|\Delta_j u_0\|_{L^2}.$$

Pour finir, prenons $\varepsilon = 2\delta$. On a alors, en utilisant (B.1),

$$\begin{aligned}
\left\|e^{-\frac{t}{2}}U^+(t)u_0\right\|_{L^2_t L^\infty} &\leq \sum_{j\geq -1}\left\|e^{-\frac{t}{2}}U^+(t)\Delta_j u_0\right\|_{L^2_t L^\infty}\\
&\leq C\sum_{j\geq -1}2^{(1+\delta)j}\|\Delta_j u_0\|_{L^2}\\
&\leq C\|u_0\|_{\dot{H}^{1+\delta}}.
\end{aligned}$$

L'estimation de Strichartz est ainsi démontrée sous réserve que l'on démontre l'inégalité de dispersion. □

Démontrons maintenant l'estimée de dispersion utilisée pour la preuve du Lemme A.1.

Lemme A.2 (Estimée de dispersion)**.** *Soient $j \geq -1$ et $t > 0$. Alors, pour toute fonction $f \in L^1(\mathbb{R}^3)$, il existe une constante C telle que*

$$\|U^+(t)\Delta_j f\|_{L^\infty} \leq C 2^{3j} \min\Big(1, \frac{1}{2^j t}\Big) \|f\|_{L^1}\,, \tag{A.3}$$

où $\Delta_j f$ est le j-ème bloc dyadique de la décomposition de Littlewood-Paley de f (voir Annexe B).

Démonstration. Soit $\omega \in C_0^\infty(\mathbb{R}^3)$ la fonction supportée dans une couronne $\mathcal{C}$ telle que

$$\Delta_j f = \mathcal{F}^{-1}\Big(\omega\big(\frac{|\xi|}{2^j}\big)\Big) * f =: \Omega_j * f\,.$$

Il suffit donc de montrer que

$$\|U^+(t)\Omega_j(x)\|_{L^\infty} \leq C 2^{3j} \min\Big(1, \frac{1}{2^j t}\Big)$$

En utilisant la rotation $x \in \mathbb{R}^3 \mapsto (0, 0, |x|)$, on obtient

$$\begin{aligned}
\big|U^+(t)\Omega_j(x)\big| &= \frac{1}{(2\pi)^3}\left|\int_{\mathbb{R}^3} e^{i\frac{t}{2}\sqrt{4|\xi|^2-1}}(1-\varphi(\xi))\omega\left(\frac{|\xi|}{2^j}\right) e^{i|x|\xi_3}\,\mathrm{d}\xi\right| \\
&= \frac{2^{3j}}{(2\pi)^3}\left|\int_{\mathcal{C}} e^{i2^j t\sqrt{|\xi|^2-2^{-2j-2}}}(1-\varphi(2^j\xi))\omega\left(|\xi|\right) e^{i2^j|x|\xi_3}\,\mathrm{d}\xi\right| \\
&\leq \frac{2^{3j}}{(2\pi)^3}\int\left|\iint e^{i2^j t\sqrt{|\xi|^2-2^{-2j-2}}}(1-\varphi(2^j\xi))\omega\left(|\xi|\right)\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right|\mathrm{d}\xi_3
\end{aligned}$$

Il suffit donc de démontrer que

$$\forall \xi_3 \in \mathbb{R}\,,\ \forall \lambda > 0\,,\ \left|\lambda\iint e^{i\lambda\sqrt{|\xi|^2-2^{-2j-2}}}(1-\varphi(2^j\xi))\omega\left(|\xi|\right)\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right| \leq C\min(1,\lambda).$$

D'abord, remarquons que, sur le support de $1-\varphi$, on a

$$|\xi| \leq \sqrt{4|\xi|^2-1} \leq 2|\xi|.$$

De plus, si j est assez grand, on a

$$(1-\varphi(2^j\xi))\omega(\xi) = \omega(\xi)$$

pour tout $\xi \in \mathcal{C}$. Finalement, on doit donc estimer l'intégrale

$$I := \lambda\iint e^{i\lambda\sqrt{|\xi|^2-2^{-2j-2}}}\omega\left(|\xi|\right)\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

uniformément par rapport à j et ξ_3.

Si $\lambda \leq 1$, on a immédiatement $I \leq C\lambda|\mathrm{supp}\,\omega|^2$ et le lemme est démontré. Supposons maintenant que $\lambda \geq 1$ et notons

$$\sigma_j(\xi) = \sqrt{|\xi|^2 - 2^{-2j-2}}$$

pour alléger les notations. Pour estimer I, on va d'abord tronquer l'intégrale en $\xi' = (\xi_1, \xi_2)$ sur une boule de rayon $\varepsilon \leq |\mathrm{supp}\,\omega|$. Pour cela, prenons une fonction $\theta \in C_0^\infty(\mathbb{R})$ à support dans $[-1, 1]$. On a alors

$$I \;=\; \lambda \iint \theta\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi' + \lambda \iint (1-\theta)\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi'.$$

On a immédiatement

$$\left|\lambda \iint \theta\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi'\right| \leq C\lambda\varepsilon^2.$$

Il reste à montrer que

$$|\tilde{I}| := \left|\lambda \iint (1-\theta)\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi'\right| \leq \frac{C}{\lambda\varepsilon^2} \tag{A.4}$$

puis on choisira $\varepsilon = \frac{1}{\sqrt{\lambda}}$ et le lemme sera démontré.
On va obtenir l'estimation (A.4) en forçant des intégrations par parties sur $a(\xi) := e^{i\lambda\sigma_j(\xi)}$ en la variable ξ_1 puis en ξ_2. On a, pour $k = 1$ ou 2,

$$\partial_{\xi_k} a(\xi) = i\lambda \frac{\xi_k}{\sigma_j(\xi)} a(\xi).$$

On veut donc éviter que ξ_k s'annule lorsque l'on intègre par parties en ξ_k. Pour cela, on introduit les fonctions tests α_1, α_2 telles que

$$\begin{array}{l} \forall \sigma \in \mathbb{S}^1\,,\ (\alpha_1 + \alpha_2)(\sigma) = 1\,, \\ \mathrm{supp}\alpha_1 \subset \left\{\sigma \in \mathbb{R}^2 : |\sigma_1| \geq \frac{1}{3}|\sigma_2|\right\}\,, \\ \mathrm{supp}\alpha_2 \subset \left\{\sigma \in \mathbb{R}^2 : |\sigma_2| \geq \frac{1}{3}|\sigma_1|\right\}\,, \end{array}$$

On a donc

$$\begin{aligned} \tilde{I} &= \lambda \iint \alpha_1\left(\frac{\xi'}{|\xi'|}\right)(1-\theta)\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi' \\ &\quad + \lambda \iint \alpha_2\left(\frac{\xi'}{|\xi'|}\right)(1-\theta)\left(\frac{|\xi'|}{\varepsilon}\right) e^{i\lambda\sigma_j(\xi)}\omega(|\xi|)\,\mathrm{d}\xi' \\ &= I_1 + I_2\,, \end{aligned}$$

Par double intégration par parties, on a, pour $k = 1$ ou 2,

$$|I_k| \;=\; \left|\lambda \iint e^{i\lambda\sigma_j(\xi)}\partial_{\xi_k}\left(\frac{\sigma_j(\xi)}{i\lambda\xi}\partial_{\xi_k}\left(\frac{\sigma_j(\xi)}{i\lambda\xi}\alpha_k\left(\frac{\xi'}{|\xi'|}\right)(1-\theta)\left(\frac{|\xi'|}{\varepsilon}\right)\omega(|\xi|)\right)\right)\mathrm{d}\xi'\right|.$$

En utilisant que

$$\left|\partial_{\xi_k}\left(\frac{\sigma_j(\xi)}{\xi}\right)\right| \leq \frac{C}{|\xi'|}\,,\ \left|\partial_{\xi_k}\alpha_k\left(\frac{\xi'}{|\xi'|}\right)\right| \leq \frac{C}{|\xi'|}\,,\ \left|\partial_{\xi_k}\theta\left(\frac{|\xi'|}{\varepsilon}\right)\right| \leq \frac{1}{\varepsilon}\,,\ |\partial_{\xi_k}\omega(|\xi|)| \leq C\,,$$

on obtient

$$|I_k| \leq \frac{C}{\lambda}\iint_{\{\varepsilon\leq|\xi'|\leq|\operatorname{supp}\omega|\}}\left(1+\frac{1}{|\xi'|}+\frac{1}{\varepsilon|\xi'|}+\frac{1}{|\xi'|^2}+\frac{1}{\varepsilon^2}\right)\mathrm{d}\xi' \leq \frac{C}{\lambda\varepsilon^2}$$

L'estimation de dispersion (A.3) est maintenant démontrée. □

Annexe B

Sur la théorie de Littlewood-Paley

One of the main tools we use in this paper is the dyadic Littlewood-Paley decomposition. In this subsection, we briefly recall some important results. Our main references for the subject are the books by Alinhac and Gérard [1] and by Lemarié-Rieusset [28].

First, recall that the homogeneous Sobolev norm $\dot{H}^s$ writes

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi = \sum_{p\in\mathbb{Z}} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

In particular, we can estimate this norm in terms of the L^2 norm of $\hat{u}|_{\{2^p \leq |\xi| \leq 2^{p+1}\}}$ as follows :

$$\sum_{p\in\mathbb{Z}} 2^{2ps} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \leq \|u\|_{\dot{H}^s}^2 \leq 2^{2s} \sum_{p\in\mathbb{Z}} 2^{2ps} \int_{2^p \leq |\xi| \leq 2^{p+1}} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

Now, we shall approximate the functions $\mathbf{1}_{\{2^p \leq |\xi| \leq 2^{p+1}\}}$ by smooth ones. So let us consider a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi(\xi) = 0$ if $|\xi| \geq 1 + \varepsilon$. Then define the function

$$\psi : \xi \mapsto \varphi\left(\frac{\xi}{2}\right) - \varphi(\xi)$$

which is supported in the annulus $\{1 \leq |\xi| \leq 2(1+\varepsilon)\}$.
For all tempered distribution u, that is $u \in \mathcal{S}'(\mathbb{R}^d)$, we define $\Delta_p u$ the p^{th} dyadic block of u by

$$\widehat{\Delta_p u}(\xi) = \psi(2^{-p}\xi)\hat{u}(\xi)$$

and we denote the partial sums by

$$S_p u = \sum_{q<p} \Delta_q u$$

for all $p \in \mathbb{Z}$.
Now, notice that $\forall \xi \neq 0$, $\sum_{p\in\mathbb{Z}} \psi(2^{-p}\xi) = 1$. So, in particular for all the functions u considered in this work, we have

$$\forall \xi \neq 0 \;,\quad \hat{u}(\xi) = \sum_{p\in\mathbb{Z}} \widehat{\Delta_p u}(\xi).$$

Remark. *Every dyadic block is in $H^{+\infty}$ once $u \in H^s$.*

We recall here some important properties of this dyadic decomposition. The results are stated for homogeneous Sobolev spaces but hold also for classical Sobolev spaces.

– (Almost-orthogonality) For all $u \in L^2$,

$$\sum_{p\in\mathbb{Z}} \|\Delta_p u\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq 2 \sum_{p\in\mathbb{Z}} \|\Delta_p u\|_{L^2}^2 .$$

– There exists a constant C such that for all $s \in \mathbb{R}$ and $p \in \mathbb{Z}$,

$$\frac{1}{C} \sum_{p\in\mathbb{Z}} 2^{2ps} \|\Delta_p u\|_{L^2}^2 \leq \|u\|_{\dot{H}^s}^2 \leq C \sum_{p\in\mathbb{Z}} 2^{2ps} \|\Delta_p u\|_{L^2}^2 . \tag{B.1}$$

We will use this property several times in this work.

– There exists a constant C such that, for all $\alpha \in \mathbb{N}^d$ and $p \in \mathbb{Z}$,

$$\begin{array}{lll} \|\partial^\alpha \Delta_p u\|_{L^2} \leq C 2^{p|\alpha|} \|u\|_{L^2} & , & \|\partial^\alpha S_p u\|_{L^2} \leq C 2^{p|\alpha|} \|u\|_{L^2}, \\ \|\partial^\alpha \Delta_p u\|_{L^\infty} \leq C 2^{p|\alpha|} \|u\|_{L^\infty} & , & \|\partial^\alpha S_p u\|_{L^\infty} \leq C 2^{p|\alpha|} \|u\|_{L^\infty}. \end{array}$$

Using this theory, we can prove some important product estimates. The following lemma tells how to write a product of tempered distributions :

Lemma B.1 (Paraproduct)**.** *Given two tempered distributions $u, v \in \mathcal{S}'(\mathbb{R}^d)$, we can write their product, if it exists, as follows :*

$$uv = \sum_{p\in\mathbb{Z}} \Delta_p u \, S_{p+1} v + \sum_{q\in\mathbb{Z}} \Delta_q v \, S_q u.$$

Then, the following proposition is very useful to estimate some products.

Proposition 1. *Let $s > 0$ and $u, v \in L^\infty \cap \dot{H}^s$. Then the product uv is also in $L^\infty \cap \dot{H}^s$ and*

$$\|uv\|_{L^\infty \cap \dot{H}^s} \leq C \left(\|u\|_{L^\infty} \|v\|_{\dot{H}^s} + \|u\|_{\dot{H}^s} \|v\|_{L^\infty} \right).$$

For a detailed proof of this proposition, see the book by Alinhac and Gérard. In this work, we also use Bernstein inequalities :

Proposition 2. *There exists a positive constant C_0 such that, for all $u \in \mathcal{S}'(\mathbb{R}^d)$, we have*

$$\begin{cases} \forall k \geq 0,\ \forall p \geq 1, \quad 2^{qk} C_0^{-k} \|\Delta_q u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^k \Delta_q u\|_{L^p} \leq 2^{qk} C_0^k \|\Delta_q u\|_{L^p} \\ \forall p' \geq p \geq 1, \quad \|\Delta_q u\|_{L^{p'}} \leq C_0 2^{dq\left(\frac{1}{p}-\frac{1}{p'}\right)} \|\Delta_q u\|_{L^p} \end{cases}$$

and, for the partial sums :

$$\begin{cases} \forall k \geq 0,\ \forall p \geq 1, \quad \sup_{|\alpha|=k} \|\partial^k S_q u\|_{L^p} \leq 2^{qk} C_0^k \|S_q u\|_{L^p} \\ \forall p' \geq p \geq 1, \quad \|S_q u\|_{L^{p'}} \leq C_0 2^{dq\left(\frac{1}{p}-\frac{1}{p'}\right)} \|S_q u\|_{L^p} \end{cases}$$

Using the dyadic blocks, we can easily define the Besov spaces $B^s_{p,r}(\mathbb{R}^d)$, where $s \in \mathbb{R}$, $p, r \geq 1$ and $d \geq 1$,

$$B^s_{p,r}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|u\|_{B^s_{p,r}} := \left(\sum_{j \geq -1} 2^{jsr} \|\Delta_j u\|^r_{L^p} \right)^{\frac{1}{r}} \right\}.$$

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